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A STUDY OF THE CONSTELLATIONS BASED ON ERATOSTHENES' CATASTERISMI

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ABSTRACT

Constellations are a distribution of the visible with the naked-eye stars in various groups. The well-known names of these groups as well as their description on sky have come from the rescued literary of the ancient Greek astronomers. In the poem 'Phenomena and Diosemia', Aratus preserves these constellations which the same, as today, names, based on the lost astronomical work of Eudoxus of Cnidus named 'Enoptron and Phenomena'. Also, in the work 'Catasterismi' by Eratosthenes is given the number of stars for each of these constellations and the corresponding myth. In this work, we study the distribution of these constellations in the sky and the distribution of stars which described these constellations. We found that the presentation of the constellations is followed five routes on the celestial sphere. Four of them are sliding from the North ending up on the Zodiac/Ecliptic and the fifth route extends almost southward of the Zodiac and East-West of the brightest star of the sky, Sirius. We also classified these constellations in groups according to the number of their stars and we discussed them in comparison with some well known, from the antiquity, 'characteristic' Pythagorean numbers and the music harmony, as this is expressed through numerical analogies of known Pythagorean numbers. We note that Eudoxus of Cnidus studied at the Pythagorean school of Archytas from Tarentum and Eratosthenes in his work named 'Platonikos' examines the mathematical foundation of Plato's philosophies (he was taught Mathematics by Theodorus of Cyrene which belonged to the philosophical school of Pythagoras), which means that he has also the knowledge of the Pythagorean theories.

KEYWORDS: Constellations, star numbers, Pythagorean numbers, history of Astronomy

1. INTRODUCTION

Constellation is a group of celestial bodies, mostly stars, that seems to sketch a formation in the sky. Twelve of such formations are called S*igns of the Zodiac* and outline the apparent path of Sun on Earth's celestial sphere; that is, *the Ecliptic or Zodiac cycle*. The projection of the Sun's orbit, as it appears from planet Earth, is connected with some of the signs of Zodiac through the beginning of the Four Seasons of a calendar year. These signs of Zodiac have been characterized as *Equinoxes* (vernal or autumnal) and (summer or winter) *Solstices*. It is necessary to point out that all these stars of the constellations that *are projected* on the celestial sphere, sketching various figures, are at different distances from Earth. Thus, a small star may look brighter than another, only because it is relatively close to Earth. In contrast, a bigger and brighter star may look dim due to the vast distance from Earth. Consequently, the stars that constitute the constellations are not necessarily close to each other. A constellation is just *a sketch of bright stars*, as they appear in the eyes of a terrestrial observer.

It is known that due to the phenomenon of the a*xial precession*, even for the positions of 'fixed' celestial bodies, changes are taking place over time. This happens because the 'Axis of the Cosmos', which is the extension of the axis of rotation of the Earth, makes a full circle like a spinning top, on the celestial sphere. This rotation lasts about 26000 years. This means that as thousands of years go by, the total picture of the celestial sphere, in the eyes of a terrestrial observer, changes. This phenomenon called a*xial precession* or the sliding of the celestial North Pole by 1 degree every 72 years. This also means that the constellations of any equinox or solstice are changing. The whole Zodiac 'belt' is rotating by one sign every 2160 years.

The first catalogue with the description of stars was created by Hipparchus (2nd century B.C.) but unfortunately was lost. Though, a catalogue of Claudius Ptolemy (2nd century A.C.) exists and it is included in his work named 'Mathematical Treatise' (13 volumes), also known as *Almagest*. In *Almagest,* 48 constellations are described, 21 to the north, 15 to the south, 12 for the Zodiac, with a total of about 1000 stars; In addition he mentions 95 amorphous stars and 18 other celestial bodies (e.g. star clusters, nebulas). However, going back in time to an earlier era, it is noted that *the names of the constellations*, as we know them today, are recorded in the poem of Aratus (305-240 B.C.) named "*Phenomena and Diosimia*" (Callimachus, Hymns and Epigrams, Lycophron, Aratus, Translated by Mair, A. W. and G. R. Loeb Classical Library Volume 129, London, William Heinemann, 1921). Aratus spent almost all his life in the court of the king of Macedonia, Antigonus II Gonatas, and it is known that he had been ordered by the latter to create such a poem. Few years after, Eratosthenes from Cyrene (276-272 to 196-192 B.C.), who lived in Alexandria during the Ptolemaic dynasty, wrote the astronomical work named "*Catasterismi*" which thankfully has been preserved in an epitome assembled at the end of the 1st century CE (the printing of *Catasterismi*, invariably attributed to Eratosthenes, began early, during the Renaissance. In this work, we used the text of *Pseudo-Eratosthenis Catasterismi* (Leipzig, 1897; Hannah, 2015; Rousseau & Dimitrakoudis, 2006).

In this specific work, Eratosthenes recorded *the same constellations* that Aratus wrote about almost 100 years before, but also gave a detailed description of the shape and *the number of the stars* that form each constellation. In addition, "*Catasterismi"* included the ancient myths that were associated with the forms of these constellations. Eratosthenes and Aratus were students of Callimachus from Cyrene (310-240 B.C.) and Eratosthenes succeeded Callimachus as Head of the Library of Alexandria. Also, both Eratosthenes and Aratus, studied in Athens, in the school of Stoicism under Zeno of Elea (336-264 B.C.). But Eratosthenes was also a student of Aristo of Chios, a cynical philosopher and a former stoic, and he also studied in the Academy of Plato; his teacher was Arcesilaus from Pitani (315-240 B.C.). In his work called "*Platonikos*", Eratosthenes examines the mathematical foundation of Plato's philosophies which are based on the theories of the Pythagorean philosophers.

However, it is well known that the poem "*Phenomena*" of Aratus (and by extension, "*Catasterismi*" of Eratosthenes) are based on the *non-preserved* astronomical work of Eudoxus of Cnidus (408-355 B.C.) named "*Enoptron and Phenomena*" in which a description of the celestial sphere had been made via the constellations. This information is given by Hipparchus in his work entitled 'Commentary to Aratus'. Eudoxus of Cnidus is considered the *Father of the Celestial Mechanics*. He developed the *Theory of Proportions*, which bridged the gap between the symmetrical and asymmetric mathematical concepts. He studied three years at the Pythagorean school of Archytas from Tarentum (428-347 B.C.) where he met and developed close relations with Plato. Also, Eudoxus stayed in Egypt for 16 months, where he studied near the priesthood of Ilioupolis (in which Pythagoras studied too). He was an excellent observer of the sky, having his very own observatories at Cnidus and at Nile Delta. He is the one who defined the positions of stars and identified the constellations through which the main circles of the celestial sphere are passing through. He grouped, located

and defined the stars that sketch forms in the sky; he named these forms and then he connected them with myths related to heroes or important events, so they could easily remain in the people's memory. In conclusion, from Eudoxus we have inherited a celestial map that was preserved through the works of Aratus and Eratosthenes.

From the previous analysis, it becomes clear that Eudoxus and Eratosthenes have taken the knowledge of the Pythagorean theories. In this work, we study the distribution of the constellations in the sky and the distribution of the star numbers in these constellations based on the work 'Catasterismi'. We also discuss these star numbers in comparison with some well-known Pythagorean numbers and the music harmony, as this is expressed through numerical analogies of known Pythagorean numbers.

2. DISTRIBUTION OF CONSTELLATIONS ON THE SKY, ACCORDING TO ARATUS AND ERATOSTHENES

The *sequence* that the constellations are presented, both in Aratus' poem and Eratosthenis' work, *is identical*. Although it is not mentioned anywhere, it is clear that five routes have been followed on the celestial sphere. Four of them are sliding from the North of the celestial sphere (the extension of the Earth's North Pole) ending up on the Zodiac/Ecliptic. The fifth route extends almost southward of the Zodiac and East-West of the brightest star of the sky, Sirius. These five described routes presented in the Figures 1, 2, 3, 4 and 5:

A) Starting at the North Pole with the Ursa Major, Ursa Minor and the constellation of Draco, the description continues with the constellation of Hercules and Corona Borealis, over Ophiuchus and Serpens, ending at the Zodiac, with the constellation of Scorpius (Fig. 1).

B) The second route begins from Bootes and continues towards the Zodiac belt, with the constellation of Virgo. After that, this description route is moving on to the Zodiac, with a different direction though; from the constellation of Gemini, to Cancer and Leo then towards the Gemini constellation, crosses over Coma Berenices, to conclude with the description of the Auriga and Taurus constellation (Fig. 2).

C) The third description route begins with the constellation of Cepheus and moving through Cassiopeia, Andromeda and Pegasus, ending at the Zodiac too, with the constellations of Aries, Triangulus (out of the Zodiac) and Pisces (Fig. 3).

D) The fourth description route initiates with the constellations of Lyra and Cygnus; continues with Aquarius and Water, Capricornus and Sagittarius which are located on the Zodiac Cycle. Following are the descriptions of Sagitta, Aquila and Delphinus; these constellations are all located outside the Zodiac (Fig. 4).

E) In the fifth group are included constellations which are stretched across near the Zodiac. The description route initiates beneath Taurus, moving towards Aquarius and ends beneath the constellation of Scorpius. Specifically, initiating from the Orion, the route continues with Canis Major and with the constellations of Lepus and Argo Navis (Puppis, Carina and Vela) extends towards east and west respectively (Fig. 6). Furthermore, the description continues east from Canis Major, including the constellations of Cetus, Eridanus and Piscis Austrinus. Then, moving west from Canis Major, the constellations of Ara, Centaurus and Lupus, Hydra, Crater and Corvus are described. In summary, the description given is based on the Canis Major constellation, which includes the brightest star of the sky, Sirius. The last described constellation of this group is Canis Minor, which is located close to Canis Major.

The total number of constellations in the description of Eratosthenes is 42 (Table 1). Though, on some occasions, two or three constellations are described together and considered as one. By dividing these constellation groups, their total number becomes 48. Despite the fact that the nomenclature of the constellations, which were given by Aratus and Eratosthenes, has been preserved through all these years, some differences have occurred. Hence, the constellation of Pegasus was once called *Hippos* (*Horse*), Piscis Austrinus was called *Piscis Magnus*, Cygnus was called *Ornis (bird) Magnus* and *Hercules* was called '*En Gonasin'* (the kneeling one). Eratosthenes refer to 'the kneeling one' as Hercules; though Aratus states that '*no one knows who he is*'.

However, there are some difference between these constellations and the 48 constellations described by Ptolemy. The constellation of Libra is not included in the signs of Zodiac by Eratosthenes, as its name was given later on. In ancient times, Libra corresponded to the *'Claws'* of the constellation of Scorpius. The star cluster of Pleiades is defined as a constellation by Eratosthenes, while today the star cluster of Pleiades is included in the constellation of Taurus. Coma Berenices is not included in Aratus' poem neither the Ptolemy's catalogue. Eratosthenes is described this constellation together with the constellation of Leo. In Ptolemy's catalogue, Aquarius constellation includes the Water constellation, while the Corona Australis constellation is separated from the Sagittarius constellation. Also Ptolemy introduced a new constellation named Equuleus.

All the constellations can be placed in a Table with 5 columns (by the number of the 5 routes). Every single 'cell' of this Table 1 would correspond to a single constellation, except for the cases where Era-

tosthenes considers some constellations as a unity. We note that the constellations are listed in this Table, in the '*same and common*' order of priority presented by Aratus and Eratosthenes. Near to the center of the Table 1 (third column, fourth row) there is the constellation of Pegasus. According to one mythological approach, the powder of the horse's galloping forms the galactic zone, which is bounded by Gemini and Sagittarius (Zodiac Equinoxes at ~6000-4000 B.C.). These two constellations are diagonal from Pegasus, in Table 1. The other diagonal couple (in relation to Pegasus) is Aquarius and Leo (Zodiac Solstices at ~4000-2000 B.C.). Additionally, Pegasus is between Cancer and Capricornus (Zodiac Equinoxes at ~2000 B.C. - 1 A.D.), while below him is Aries (Zodiac Spring Equinox at ~2000 B.C.-1 A.D.). Therefore, near to the center of the Table is the *Triangle* of the 'characteristic' zodiacs at ~2000 B.C.-1 A.D. (Cancer, Aries and Capricornus) framed by a *Quadrangle* with diagonals formed by Gemini- Sagittarius (Equinoxes at ~6000-4000 B.C.) and Aquarius-Leo (Solstices at ~4000-2000 B.C.). We note that the Scorpio (and Claws) constellation remained in the Autumn Equinox from ~4000 B.C. to 600 A.D.

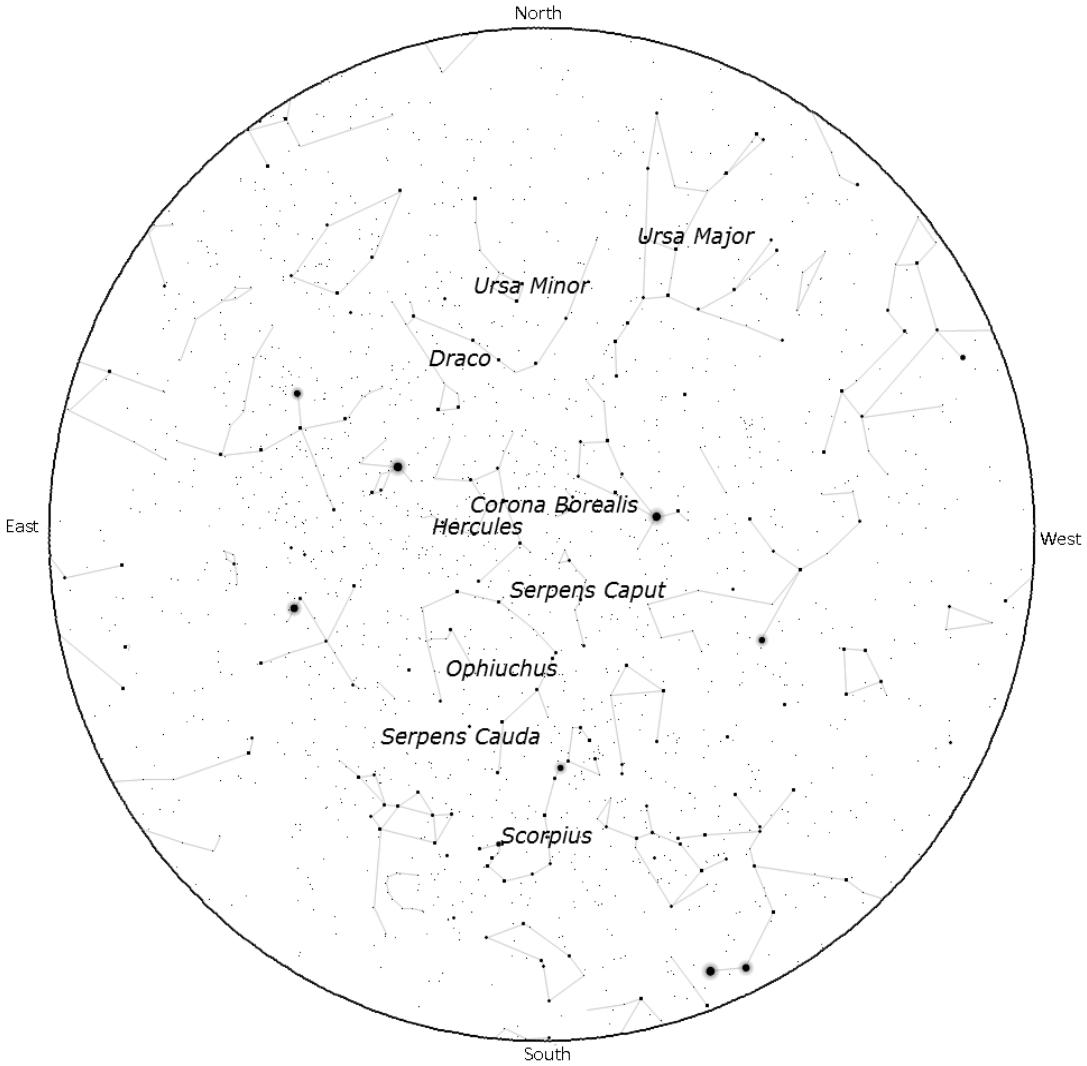

Figure 1. Celestial map, on 20 March 350 B.C., at 03:00 a.m. as seen from Alexandria, Egypt. Constellations of the first route are marked, together with the four signs of horizon.

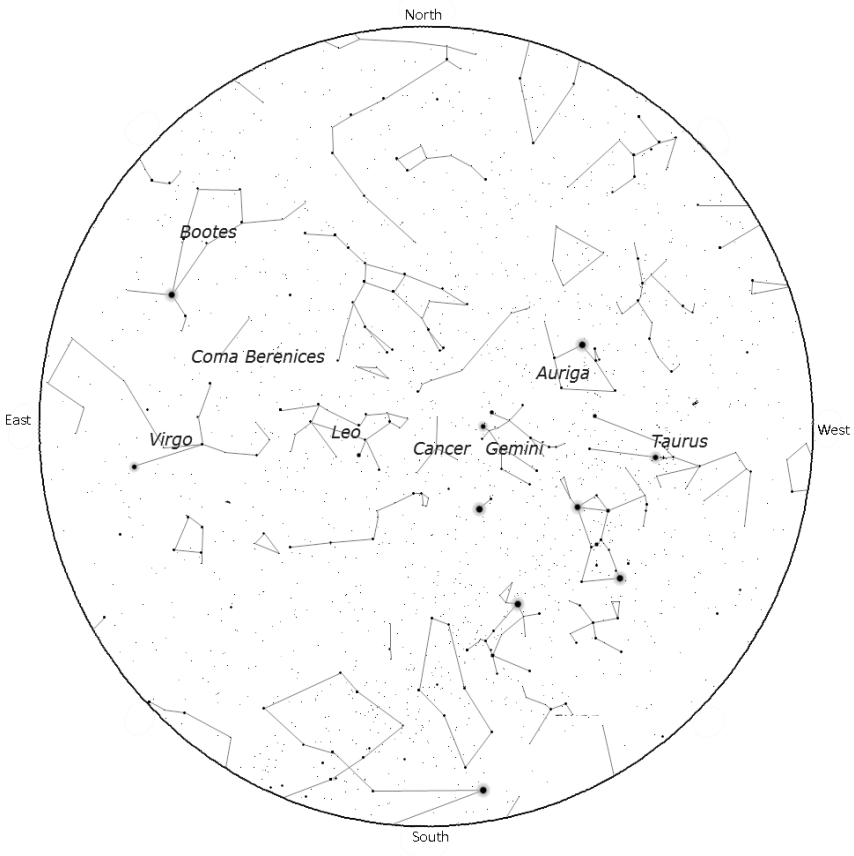

Figure 2. Celestial map, on 20 November 350 B.C., at 03:00 a.m. as seen from Alexandria, Egypt. Constellations of the second route are marked, together with the four signs of horizon.

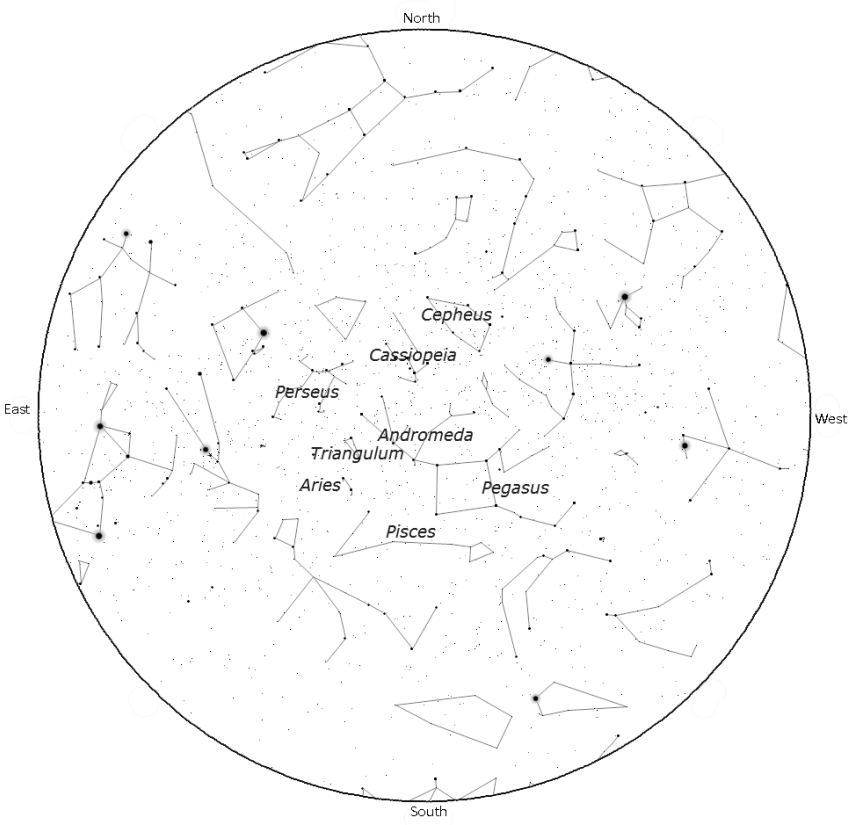

Figure 3. Celestial map, on 20 July 350 B.C., at 03:00 a.m. as seen from Alexandria, Egypt. Constellations of the third route are marked, together with the four signs of horizon.

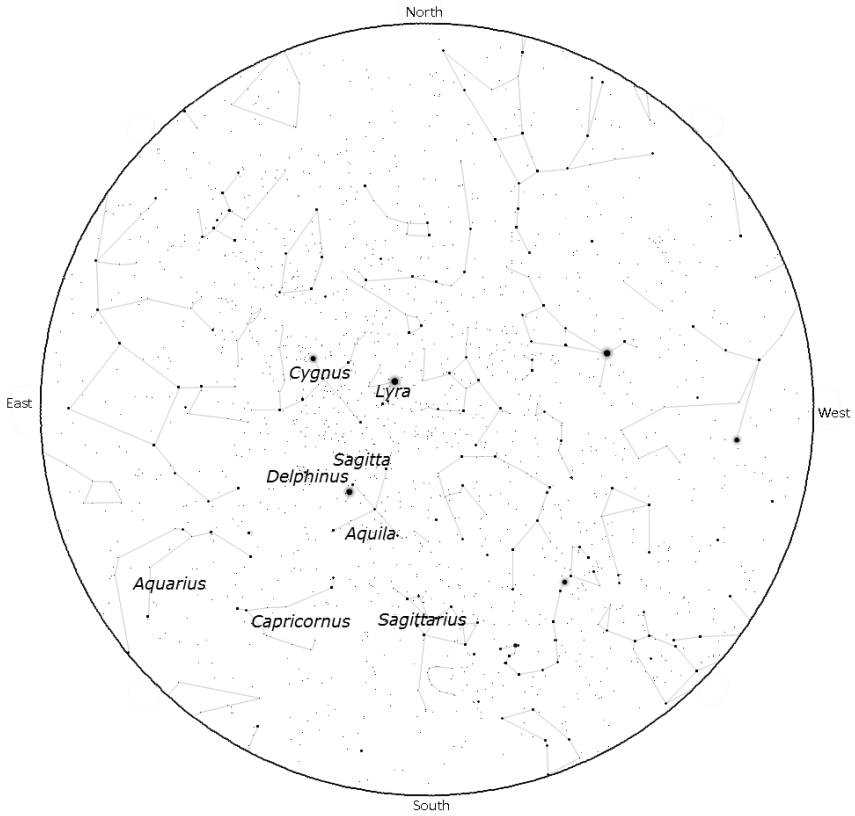

Figure 4. Celestial map, on 20 April 350 B.C., at 03:00 a.m. as seen from Alexandria, Egypt. Constellations of the fourth route are marked, together with the four signs of horizon.

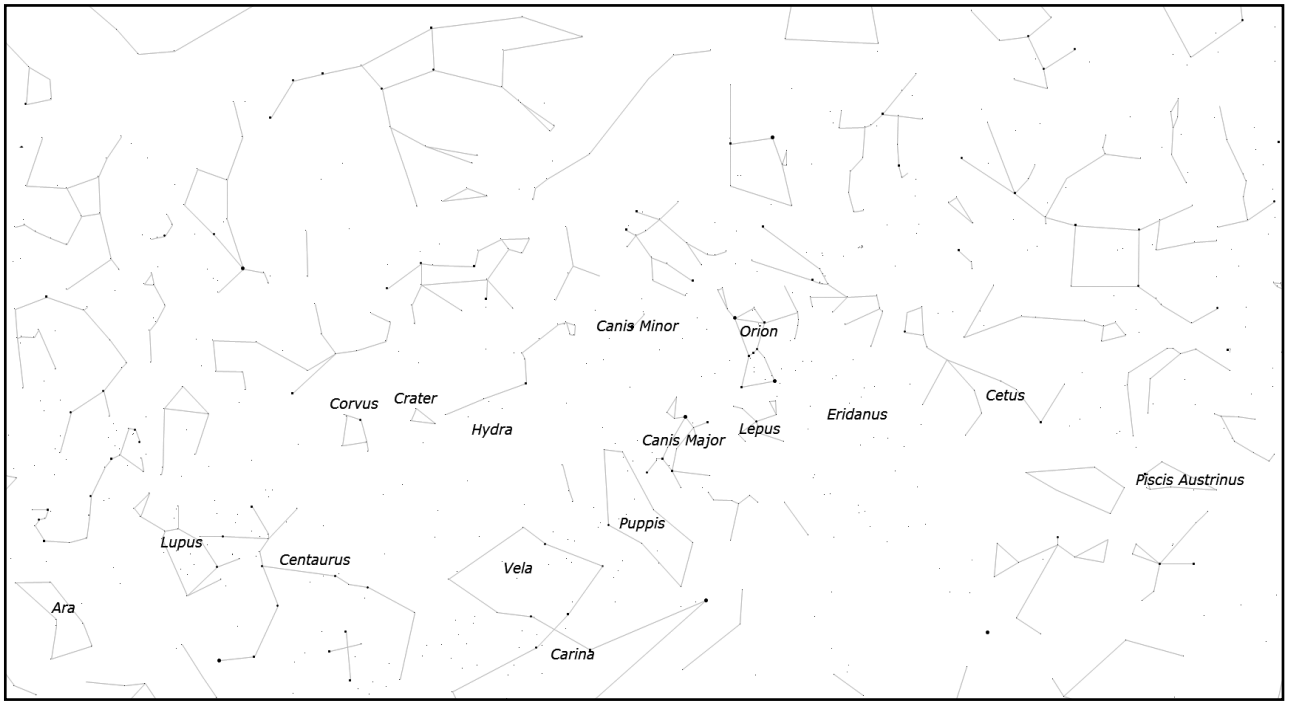

Figure 5. Celestial map for Alexandria, Egypt, marking the constellations of the fifth route

Table 1: The Constellations and their numbers of stars, according to Eratosthenes 'Catasterismi'.

First Route	Second Route	Third Route	Fourth Route	Fifth Route
Ursa Major 24	*Bootes 14*	*Cepheus 19*	*Lyra 8*	*Orion 17*
Ursa Minor 8	*Virgo 20*	*Cassiopeia 15*	*Cygnus (Bird Magnum) 14*	*Canis Major 20*
Draco 15	*Gemini 10+9=19*	*Andromeda 20*	*Aquarius and Water 17+31=48*	*Lepus 7*
Hercules (the Kneeling) 19	*Cancer 18*	*Pegasus (Hippus) 18*	*Capricornus (Pan) 24*	*Argo Navis 27*
Corona Borealis (Corona) 9	*Leo 19* *Coma Berenices (Coma or knit hair) 7*	*Aries 17*	*Sagittarius 22=15+7 (Corona Australis with 7 stars is included)*	*Cetus 13*
Ophiuchus and Serpens 17+2=19	*Auriga 8*	*Triangulum (deltoton-letter delta) 3*	*Sagitta 4*	*Eridanus 13*
Scorpius 19	*Taurus 18*	*Pisces 39 (12+12+15)*	*Aquila 4*	*Pisces Austrinus (PiscesMagnum) 12*
		Perseus 19	*Delphinus 9*	*Ara 4*
		Pleiades 7		*Centaurus and Lupus 24+10=34*
				Hydra and Crater and Corvus 27+10+7=44
				Canis Minor 3

Table 2: The constellations's number of stars, according to Eratosthenes, Hipparchus and Ptolemy

Constellation	**Eratosthenes**	**Hipparchus**	**Ptolemy**
Ursa Major	24	24	27
Ursa Minor	7+1=8	7	7
Draco	15	15	31
Hercules	19	24	28
Corona Borealis	9	9	8
Ophiuchus+ Serpens	17+2=19	17	24+18=42
Scorpius+Libra	19	15+4=19	21+8=29
Bootes	14	19	22
Virgo	20	19	26
Gemini	19	19	18
Cancer	18	16	9
Leo+ Coma Berenices	19+7=26	19	27
Auriga	8	8	14
Taurus (without Pleiades)	18	18	32
Cepheus	19	19	11
Cassiopeia	15	14	13
Andromeda	20	20	23
Pegasus	18	18	20
Aries	17	17	13
Triangulum	3	3	4
Pisces	39	41	34
Perseus	19	19	26
Pleiades	7	-	-
Lyra	8	10	10
Cygnus	14	14	17
Aquarius+Water	17+31=48	18	42
Capricornus	24	26	28
Sagittarius+ Corona Australis	15+7=22	16+?	31+13
Sagitta	4	4	5
Aquila	4	4	9
Delphinus	9	9	10
Orion	17	18	38
Canis Major	20	21	18
Lepus	7	?	12
Argo	27	13	45
Cetus	13	14	22
Eridanus	13	?	34

Piscis Austrinus	12	12	11
Ara	4	4	7
Centaurus+Lupus	24+10=34	26+13	37+19
Hydra+Crater+Corvus	27+10+7=44	27+10+7	25+7+7
Canis Minor	3	3	2
Equuleus	-	-	4
SUM	**720**	**761**	**918**

3. THE DISTRIBUTION OF THE STARS WITHIN THE CONSTELLATIONS

Eratosthenes reports the exact number of stars in each constellation that he describes. If we sum up all these stars, we get the sum of 720. The distribution of these stars in the constellations of the 5 columns of the Table 1 is as follows:

1st group (7 constellations):
24+8+15+19+9+(17+2)+19=113
2nd group (7 constellations):
14+20+19+18+(19+7)+8+18=123
3rd group (9 constellations):
19+15+20+18+17+3+39+19+7=157
4th group (8 constellations):
8+14+(17+31)+24+22+4+4+9=133
5th group (11 constellations):
17+20+7+27+13+13+12+4+(24+10)+(27+10+7)+3=194

We note that, the total number of stars, 720, is divides to 156 polar and their mythological related constellations' stars (see in the session 5), 282 zodiac's constellations stars and 282 other constellations' stars.

These stars distribute to the constellations covering a wild field between the number 2 and the number 48 (see Table 1). However, the numbers 48, 44 and 34 refer to a group of two or three constellations. Consequently, the upper limit of this field of numbers is the number 39 (the number of stars of the constellation Pisces) and the lower limit is the number 2 (the number of stars of the constellation Serpens). The most of these star numbers are included between 2 and 20. Only five star numbers (22, 24, 27, 31, 39) are upper than the number 20. We notice that, in this classification, until the number 20, there is the absence of some numbers. These numbers are: 1, 5, 6, 11 and 16.

We can classify the constellations in groups based on the number of their stars:

- Group of number 2 and its exponents (2, 2^2, 2^3):
  Serpens (2)
  Aquila, Sagitta, Ara (4)
  Auriga, Lyra, Ursa Minor (8)
- Group of number 3 and its exponents (3, 3^2, 3^3):
  Triangulum, Canis Minor (3)
  Delphinus, Corona Borealis, one of the "Geminies" (9)
  Argo Navis, Hydra (27)
- Group of the numbers 18 ($2^1 \times 3^2$)
  Pegasus, Cancer, Taurus (18)
- Group of the number 12 and its multiples ($2^x \times 3^1$, x=2,3,4):
  Piscis Austrinus, One of the Pisces and the Cord that connects them (12)
  Ursa Major, Capricornus, Centaurus (24)
  Aquarius and Hydria with water (48)
- Group of multiples of the number 5:
  Crater, Lupus, One of the Geminies ($10=2^1 \times 5^1$)
  Cassiopeia, Draco, Sagittarius (*without Corona Australis*), One of the Pisces ($15=3^1 \times 5^1$)
  Virgo, Canis Major, Andromeda ($20=2^2 \times 5^1$)
- Group of number 7 and the double it:
  Pleiades, Lepus, Coma Berenices, Corvus, Corona Australis (*inside Sagittarius*), (7)
  Bootes, Cygnus ($14=2^1 \times 7^1$)
- Group of multiples of the number 11:
  Sagittarius ($22=2^1 \times 11^1$)
  Hydra + Crater + Corvus ($44=2^2 \times 11^1$)
- Group of the number 13 and its triple:
  Cetus, Eridanus (13)
  Pisces ($39=3^1 \times 13^1$)
- Group of the number 17 and the double it:
  Aquarius, Ophiuchus, Aries, Orion (17)
  Centaurus + Lupus ($34=2^1 \times 17^1$)
- Group of the number 19:
  Cepheus, En Gonasin/Hercules, Leo, Scorpius, Perseus, Gemini, (Ophiuchus+Serpens), (19)

We mention the important difference between the 720 stars of Eratosthenes' description of constellations and the corresponding 918 stars of Ptolemy's catalogue. It is known that from the 1025 stars which are included in the Ptolemy's catalogue, only 918 stars (Table 2) are inside the constellations and the others characterized as 'informs' stars. Also, there is an opinion that these 1025 stars of the Ptolemy's catalogue must have occurred in the lost catalogue of Hipparchus too.

However, in 1892 there was first brought the light a hitherto unknown Greek list of the constellations attributed to Hipparchus and since then several copies of it have been found, the best one being part of Cod. Parisinus 2506 (Grec 2506, Bibliotheque Nationale in Paris), entitled 'εκ των υπάρχων αστέρων' (From the stars of Hipparchus). It has been discussed by F. Boll in a paper published in 1901 (Die SternKataloge des Hipparch und des Ptolemaios, in the Bibliotheca mathematica, 3rd series, v. ii, pp. 185-195). In it simply counts up the constellations and adds to each a num-

ber, being the number of stars in it (Table 2). This information includes in the paper entitled 'On the origin of Ptolemy's catalogue of stars' by J. L .E Dreyer in Monthly Notices of the Royal Astronomical Society, v. 77, pp. 528-539, May 1917.

According to this Hipparchus' catalogue, in 43 from 46 constellations the number of stars is given. In 30 cases, the numbers are exactly the same as in the Eratosthenes' Catasterismi. The only serious exception is the stars number of Argo constellation (13 and 27 stars respectively). In the other 12 cases only one, Cancer, has a smaller number that the Catasterismi (16 and 18 respectively), while in the remaining 11 cases (Bootes, Hercules, Lyra, Centaurus, Lupus, Cetus, Orion, Canis Major, Sagittarius, Carpicornus and Aquarius) Hipparchus has from one to five stars more than Eratosthenes. However, in most cases the difference is only one to two stars (Table 2).

It is remarkable that the number of stars which described the constellations according to Ptolemy is greater than the corresponding number of stars which given by Hipparchus or Eratosthenes. Additionally, this number of stars is almost the same, except of few cases, in Hipparchus and Eratosthenes catalogues.

4. DISTRIBUTION OF STARS IN THE CONSTELLATIONS OF EQUINOXES AND THE SOLSTICES

It is known that Hipparchus is generally recognized as discoverer of the precession of the equinoxes, in 127 B.C. although there are claims for its earlier discovery. Virtually all of the writings of Hipparchus are lost, including his work on precession, but it's mentioned by Ptolemy. However, there is still-controversial evidence that Aristarchus of Samos (310-230 B.C.) possessed distinct values for the sidereal year (the time it takes the Sun to return to a fixed star) and tropical year (the time it takes the Sun to return to an equinox), as early as 280 B.C. (Dennis Rawlins, *Continued fraction decipherment: the Aristarchan ancestry of Hipparchos' year length & precession DIO* (1999) 30–42). Moreover, various assertions have been made that other cultures discovered precession independently of Hipparchus. According to Al-Battani, the Chaldean astronomers had distinguished the tropical and sidereal year so that by approximately 330 B.C., they would have been in a position to describe precession, if inaccurately, but such claims generally are regarded as unsupported (Neugebauer, O. "*The Alleged Babylonian Discovery of the Precession of the Equinoxes*", *Journal of the American Oriental Society*, Vol. 70, No. 1. (Jan. – Mar., 1950), pp. 1–8). Similar claims have been made that precession was known in Ancient Egypt during the dynastic era, prior to the time of Hipparchus. However, these claims remain controversial (Martin Castro, 2015).

Due to the phenomenon of the axial precession, the constellations corresponding to the positions of Equinoxes and Solstices are changing every ~2000 years. In the Table 3 are notified all the constellations that were alternated in the Equinoxes and the Solstices, from ~6000-7000 B.C. until ~1 A.C. It concerns three time periods, corresponding to the position of Draco and Ursa Minor in the North Pole. We note that constellation Scorpius remains in the Autumnal Equinox over a long period of time. Also, as explained by Eratosthenes, the constellation of Sagittarius is formed by 15 stars, plus 7 more which are located beneath Sagittarius legs; afterwards those stars named as the constellation of 'Corona Australis'.

5. DISTRIBUTION OF STARS IN THE POLAR CONSTELLATIONS

There are *six polar constellations* (Ursa Minor, Draco, Hercules, Lyra, Cygnus and Cepheus) that alternate successively on the North Celestial Pole. This is a result of procession; the Celestial North Pole follows a circular pattern on the sky, once every ~26000 years. The number of stars, in order of appearance of these six constellations at the North Pole is: 8, 19, 14 and 8, 19, 15 and so on (Table 4).

Analyzing the number of stars of Ursa Minor and Draco constellations, according to Eratosthenes' description about the shape of their formations; we show that they are relevant with the numbers 3 and 4, because even the number 12 can be expressed as 3×4. Also, the number 14 (for the stars of Cygnus constellation) is the double of the number 7 which is presented as 3+4 stars for the Ursa Minor constellation. We mention that the number 19 (for the stars of Cepheus and Hercules constellation) can be expressed as the sum of the numbers 7 and 12. According to this approximation, all the stars' numbers of these constellations can be expressed as a combination of the numbers 3 and 4.

There is a mythological relation between the constellations of Ursa Major and Ursa Minor (mother and son). However, the Ursa Major constellation called also 'Wagon' and is followed by the constellation of Bootes (which means the 'carter of the ox-cart'). The sum of stars of these 'related' constellations is:

Ursa Major+Bootes+Ursa Minor = 24+14+8 = 46.

Additionally, there is a mythological relation between the constellations of Cepheus, Cassiopeia and Andromeda (parents and dotter). The sum of the stars' numbers of these three constellations is:

Cepheus+Cassiopeia+Andromeda=19+15+20=54.

Therefore, the sum of the stars of the above referred polar constellations and the ones they are related to mythologically is: $46+54=100=10^2$.

The sum of the stars of the rest constellations (Cygnus, Lyra, Hercules and Draco) is equal to 56 (7×8). Adding all the stars numbers of the above constellations we found the number 156, which analyzing in a production of numbers is equal to 4×39=4×3×13=12×13.

Table 3. The constellations of Equinoxes and Solstices and their stars number

Time period	Winter Solstice	Summer Solstice	Vernal Equinox	Autumnal Equinox
~6000 B.C. -4000 B.C.	Pisces 39 (12+12+15) (6600-4000 B.C.)	Virgo 20 (7100-4000 B.C.)	Gemini 19 (6600-4400 B.C.)	Sagittarius 22 (15+7) (6600-4000 B.C.)
~4000 B.C. - 2000 B.C.	Aquarius+Water 17+31=48 (4000-2200 B.C.)	Leo 19 (4000-1300 B.C.)	Taurus 18 (4400-1800 B.C.)	Scorpius and Claws 19 (4000-2200 B.C.)
~2000 B.C. - 1 A.D.	Capricornus 24 (2200 B.C.-100A.D.)	Cancer 18 (1300 B.C.-1 A.D.)	Aries 17 (1800 B.C.-1 A.D.)	Scorpius and Claws 19 (2200 B.C.-600 A.D.)

Table 4. The Polar Constellations and their stars number

Constellation	Number of stars	Chronology on the North Pole
Ursa Minor* (Polaris)	8=7+1= (4+3)+1	25500-22000 B.C.
Cepheus	19=7+12=(4+3)+(4×3)	22000-16000 B.C.
Cycnus (Big Bird)	14=2×7=(4+3)+(4+3)	16000-13000 B.C.
Lyra	8=4+4	13000-11000 B.C.
Hercules (The kneeling one)	19=7+12=(4+3)+(4×3)	11000-6000 B.C.
Draco**	15=3+12 =3 + (4×3)	6000-500 B.C.
Ursa Minor* (Polaris)	8=7+1= (4+3)+1	500 π.Χ. -3500 B.C.

**This constellation has 1 bright star on each angle of the quadrilateral and 3 more on its tail, totally 7 stars. Beneath those stars, there is another one called Polaris, around which the Cosmos revolved. (Eratosthenes Catasterismoi 1, 2, 16-20).*

*** This constellation has 3 stars on its head. On his body and tail have 12 more stars, close to each other and far away from Ursa constellations. (Eratosthenes Catasterismoi, 1, 3, 16-19).*

6. IS THERE ANY RELATION BETWEEN THESE STARS' NUMBERS AND THE PYTHAGOREAN THEORY?

6.1 The stars' numbers and the Pythagorean Theory of numbers

Pythagoras assumed that everything in the world is '*numbers*', having learned from the Egyptians and the Babylonians that numeric values had been attributed to the planets and stars and having discovered that *music* could be expressed through arithmetic relations. And, of course, both stars and planets must have been musical beings, since both of them are numbers and numbers are the basis of music. This is the main idea of *the harmony of the celestial spheres.* Hence, Pythagoras becomes the first philosopher to combine Astronomy with Music claiming that in a harmonic and spherical universe everything is governed by simple laws that can be expressed through the numbers of the '*Holy Tetractys*'. According to Plutarch (*On Music*, Stephanus 1147, A, 7-12):''*because those associated with Pythagoras, Archytas and Plato and the rest of the ancient philosophers claimed that the bearing of beings and the movements of stars could neither happen nor be sustained without music; they say that everything has been created by God based on the laws of harmony*'. Music used to be a mathematical science for the Pythagoreans, like Arithmetic, Geometry and Astronomy. The description of the universe is included in *eight celestial spher*es, out of which seven belong to the seven *planets* (the Sun, the Moon, Mercury, Venus, Mars, Jupiter and Saturn), while the eighth one is the sphere of *fixed stars*. We note that, *this latter sphere* is outlined by the layout of *the constellations*, as it was shaped and made public by Eudoxus of Cnidus (which was a student of Archytas) in his work "*Enoptron and Phaenomena*".

It is known that the teaching of Pythagoras was secret and was carried out through the use of special codes. According to Pythagoreans, the numbers are the primary principle of Cosmogony, while the "*Holy Tetractys*" is the source of Creation. Porphyry of Tyre (*Vita Pythagorae*, 20, 14-19) reports, that the Pythagorean philosophers were obliged to take an oath in the name of Pythagoras and the Holy Tetractys, every time they wanted to affirm something. According to Nicomachus of Geras (*Introductio arithmetica* 27, 15), Tetractys is '*4 by force*' and '*10 by number*' which means that the first four natural numbers have the sum 10 (1+2+3+4=10). It is also known that the content of Tetractys is expressed by *musical relations* (Fig. 7), as Theon of Smyrna interprets in his

work '*De utilitate mathematicae*, 58, 13 – 59, 3': '*The Tetractys consisting of numbers 1, 2, 3 and 4 contains all the symphonies. These numbers express the diatessaron symphony, the diapente symphony, the diapason symphony or octave (2/1), the diapason plus diapente symphony and the disdiapason symphony.' The Pythagorean mathematical expressions of the above mentioned musical symphonies are the epitritus ratio (4/3), the hemiolius ratio (3/2), the double ratio (2/1), the triple ratio (3/1) and the fourfold ratio (4/1), respectively.*

The sum of the numbers of the squares of the Tetractys' numbers is equal to 30 which is the average number of days in a *lunar month* (29.53~30 days): $1^2+2^2+3^2+4^2 = 1+4+9+16 = 30$. Respectively, the sum of the forth power of each number of Tetractys is equal to the number 354, which is the total number of days of a *Lunar Year*:

$$1^4+2^4+3^4+4^4=1+16+81+256=354.$$

Also, the number 100 (10^2) equals with the sum of the cubes of all the Tetractys' numbers, that is:

$$1^3+2^3+3^3+4^3=1+8+27+64=100.$$

We remind that the sum of the two 'families' of the polar constellations of Ursa Minor and Cepheus as well as the sum of stars of the constellations of the solstices and equinoxes of ~6000-4000 B.C. are equal to 100 (see sessions 4 and 5). Moreover, the stars number of Crater or Lupus constellations is equal to 10. The other numbers of the Holy Tetractys are also appeared as the stars number of various constellations, except of the number 1 because is it impossible to describe a set of stars with only one point.

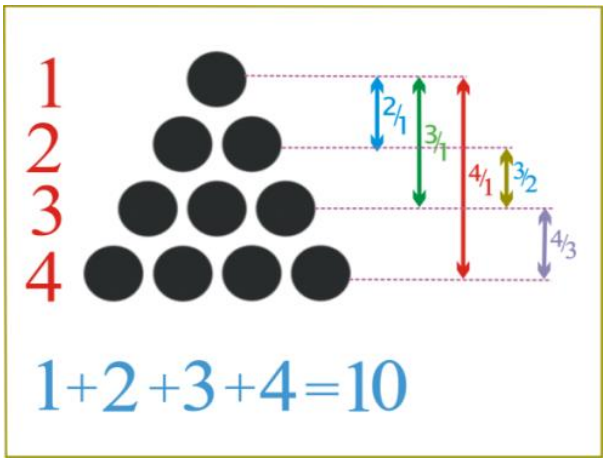

Figure 7: The triangular representation of the 'Holy Tetractys' and its musical relations. (Ch. Spyridis, The beliefs of the Pythagoreans about Harmonics, Publishing Address and Editions of the National & Capodistrian University of Athens, Athens (2014), p. 172).

Furthermore, Lucian of Samosata (*Vitarum auctio*, 4, 5-6) reports that Tetractys is geometrical expressed with a 'perfect' triangle - isosceles triangle -, for the reason that numerically is expressed by the 'triangular number' of 10=1+2+3+4. However, the *number 4*, which corresponds to the first four natural numbers included in Tetractys, is symbolized in ancient time with the G*reek letter 'delta (Δ)'*, the symbol of which is similar to an isosceles triangle. It should be noted that there is a constellation, located above Aries (see Table 1), called "*Deltoton*" by Eratosthenes and Aratus, which means the Greek '*letter Δ (delta)*' and is presented as an isosceles triangle. Today this constellation is named '*Triangulum*' (Fig. 8).

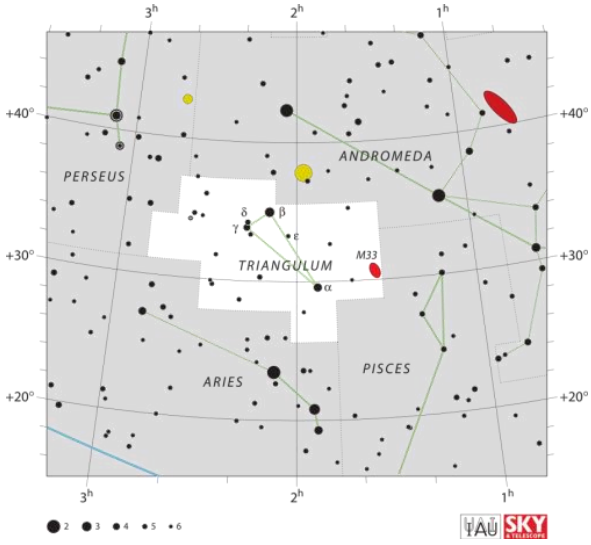

Figure 8: The Triangulum Constellation. Its old name was 'Deltoton' which translates the Greek 'letter Δ (delta)'. Also, the number 4 is symbolized in ancient time with the Greek letter 'Δ'. Source: https://upload.wikimedia.org/wikipedia/commons/thumb/8/81/Triangulum_IAU.svg/1200px-Triangulum_IAU.svg.png

The *Pythagorean theory of numbers* is a hierarchy of numbers. Specifically, the even numbers (2k, where k is a Natural Number) are considered as female numbers, while the odd numbers (2k+1, where k is a Natural Number) are considered as male numbers. The addition or the multiplication of two numbers is called 'marriage'. Also, by multiplying the numerical relations of two musical intervals, which are defined as length ratios, we calculate their 'Pythagorean sum'.

According to Iamblichus (245 – 325 A.D.) in *Theologoumena arithmeticae (30, 17) or* Anatolius, in his work '*On the Ten and the numbers therein'*, the number 5, which corresponds to the number of the constellation groups which follow the five routes on the celestial sphere (see Table 1), is an important Pythagorean number. It is the 'fertile marriage' of the first Even (*female*) entity with the first Odd (*male*) number; that is 5=3+2. This expresses the '*analog law'* for the natural numbers of the ten, as it is their average:

$$\frac{\sum_{i=1}^{9} x_i}{i} = \frac{1+2+3+4+5+6+7+8+9}{9} = \frac{45}{9} = 5$$

Moreover, Anatolius ('O*n the Ten and the numbers therein'*), and Plutarch (*De apud Delphos E*, 391, A, 1-7) mention that the number 5 is the sum of the squares of the first two numbers (1 and 2): $5=4+1=2^2+1^2$.

It is remarkable that this Pythagorean interpretation of the number 5 is appeared in these five routes of the constellations. Specifically, the four (2^2) description routes of constellations are commencing from the North Celestial Pole, ending South, on the Zodiac and the fifth route (1^2) extends in two different directions beneath the Zodiac, towards west and east, with the brightest star Sirius (or the constellation of Canis Major) at a center.

Also, Pythagoreans defined the numbers 5 and 6 as *circular or spherical numbers*, for the reason that all their exponentiations end to themselves respectively i.e. $5^2=25$, $5^3=125$, $6^2=36$, $6^3=216$ etc. The number 6, according to Pythagoreans: a) is a "perfect" number because 6/6=1, 6/3=2, 6/2=3, 1+2+3=6 (Nicomachus of Geras, *Introductio arithmetica*, 1, 16, 1, 1-6), b) is a "triangular" number (it extract from the sum 1+2+3+4+…+n, e.g. 1+2=3, 1+2+3=6, 1+2+3+4=10 etc) and c) is equal to the area of the Pythagorean triangle (see in the next paragraphs). Furthermore, for the reason that the number 6 can be analyzed in a sequence of the five fundamental harmonic musical intervals of the diatonic and the chromatic genus as well (6= (2/1) × (3/2) × (4/3) × (5/4) × (6/5) i.e. diapason, diapente, diatessaron, great chromatic third and minor chromatic sixth), this number is called the *Soul of Cosmos*, in other words the *cernel* of the chromatic musical scale with harmonic intervals (Plato, *Respublica*, 546).

Moreover, the ideal shape of Cosmos is the sphere and the '*perfect shape*' is the circle, according to Pythagoreans. The Zodiac Cycle (12 zodiac constellations) is reproduced by a *regular hexagon* inscribed in circle. But the Sun's movement across the Zodiac defines the meaning of a year (solar year), as Plato explains (*Definitiones*, 411b, 3): '*Time comes from the periodic motion of the Sun'*. According to Homeric epics, the time of the day and night was divided into *6 parts* (Eos, Middle of the Day, Dusk, Afternoon, Middle Night, Dawn).

However, the total number of stars describing these constellations is 720. That is: 720 = 6! = 1×2×3×4×5×6.

Therefore, the *total star population number* is connected numerically with the 'perfect' *number 6*, the 'Soul of Cosmos' (6, for 6!=720) and is divided in 'specific' 5 ($5=4+1=2^2+1^1$) described routes on the sky (heavenly sphere), when the ideal shape of 'Cosmos' is the sphere.

Indeed, Plutarch (1st c A.C.) introduced another form of Tetractys which he named *Cosmos*. The first four even natural numbers (2, 4, 6, 8) and the first four odd natural numbers (1, 3, 5, 7) are forming the Plutarch's Tetractys. He considered the *number 36 (6^2)* that is the sum of these eight first natural numbers, as the '*Rotating Cosmos*' *('The Tetractys, which the Pythagoreans praise, that is the number 36, is considered admirable, because it results from the sum of the first four even numbers and the first four odd numbers'* Plutarch, '*De animae procreatione in Timaeo'*, 1027 F 2-5). The circular number 6 and its exponents are symbolically indicated that Cosmos is rotating *(The so-called Tetractys that is the number 36 was a great oath, as it is legendary and was also called the world resulting from the sum of the first four even numbers and the first four odd numbers', Plutarch, 'De Iside et Osidire'*, 381F,6-382A, 4).

We mention that, the total number of stars (720=2×360= 2×10×36) that decorate the starry sky according to Eratosthenes is related with the number 36 which means '*Cosmos*' and with the number 10 ('*Holy Tetractys'*). Indeed, the Greek word 'Cosmos' is produced by the verb '*κοσμώ (cosmo)'*=adorn, decorate. Also, the symbol of Tetraktys, the isosceles triangle is appeared in this sky decoration named 'Deltoton' which means the Greek *letter Δ (delta),* the symbol of the number 4, expressed the four numbers of 'Holy Tetractys' (1+2+3+4=10).

The total number of stars described the constellations according to Hipparchus (see Table 2) is 761 plus the three unknown numbers related with three constellations. Adding their corresponding numbers from the Eratosthenes catalog, this 'Hipparchus number' replaced by the number 788. If we assumed that somebody changes the number 10 with the number 11 in the previous Eratosthenes' relation (720=2×10×36), the number of stars will be done 2×11×36=792. The small difference between 788 and 792 is perfectly compatible with the statement that in most of the cases there is a small difference in the number of stars of each constellation, according to Eratosthenes and Hipparchus.

It is known that Plato was taught Mathematics by Theodorus of Cyrene who belonged to the philosophical school of Pythagoras. The influence of the Pythagoreans upon his theories is undeniable. The Plato's Cosmogony is demonstrated in the dialogue '*Timaeus'*, where the *Greatest Platonic Tetractys* is referenced (Fig.9). This Tetractys is composed from the numbers 1, 2, 3, 4, 9, 8 and 27, which is the one (1) and the combination of two groups, one created by the number 2 and its exponents until the third power (2, $4=2^2$, $8=2^3$), and another with the number 3 and its exponents until the third power (3, $9=3^2$, $27=3^3$). This combined Tetractys symbolizes our *Three-dimensional Cosmos* (*Timaeus*, 32b, 1-2). Indeed, according to Theo of Smyrna (De utilitate mathematicae, 95, 2-13): '*Two Tetractys occur by multiplication, the 'even' and the 'odd' one. The even Tetractys results*

with double ratio because the first even number is the '2'. The number 2 results from the number 1 by doubling. The odd Tetractys results with triple ratio because the first odd number is the '3'. The number 3 results from the number 1 with tripling. So, both Tetractys, the even and odd one, have number 1 in common. The second number is 2 for the even Tetractys and 3 for the odd Tetractys. The third number is 4=2^2 for the even Tetractys and 9=3^2 for the odd Tetractys. The third number is 4=2^2 for the even Tetyractys and 9=3^2 for the odd Tetractys. The forth number is 8=2^3 for the even Tetractys and 27=3^3 for the odd Tetractys.'

We remind the reader that there are two groups in the constellations' star classification (see session 3) which are exactly the number 2 and its exponents until the third power (2, 2=2^2, 8=2^3) and the number 3 and its exponents until the third power (3, 9=3^2, 27=3^3).

Adding the numbers of the two groups of the *Greatest Tetractys*, we have the following results: 2+2^2+2^3=2+4+8=14=2×7 and 3+3^2+3^3=3+9+27=39=3×13. Also, the total sum of the numbers of the *Greatest Tetractys* is 54 (14 + 39=54 = 6×9).

However, there are two constellations (Bootes and Cygnus) which are described by 14 stars, while the stars which describe the Pisces constellation are 39. Also, we remind that the number of stars of the Cepheus 'family' is 54 and the total sum of the stars of polar constellations included the constellations of Ursa Minor's and Cepheus' 'families' is 156=4×39 (see session 5). Additionally, the total number of stars described the constellations according to Ptolemy (Table 2) is 918. However, the number 918 is equal to 17×54.

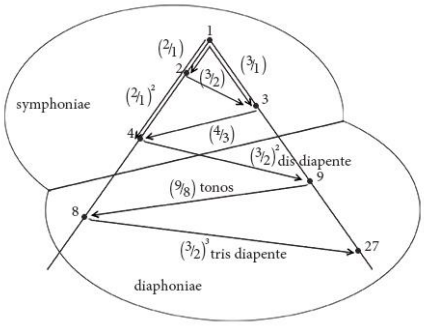

Figure 9. The Plato's Greatest Tetraktys. There is a separate line between the upper (1, 2, 3 and 4) and bottom (9, 8 and 27) part of it. The musical ratios of these two parts consist symphoniae or diaphoniae respectively.

The terms of the *Greatest Tetraktys* (Fig. 9) are separated in two parts: 1+2+3+4=10 (*Holy Tetraktys*) and 9+8+27=17+27=44, while the number 17 is equal to the sum ('marriage') 3^2+2^3. The 'connection' between these two parts is the sum (marriage') of 4 (last number of 1st part) and 9 (first number of 2nd part), 2^2 + 3^2=13. Also, the middle number of the *Greatest Tetraktys* is the number 4. The sum ('marriage') of the first and final number of this Tetraktys with its middle number is 1+4=5 and 4+27=31 respectively. We note that 31 is equal to 2^2 +3^3 which is the opposite (referred to the exponents) of 17=3^2+2^3. The sum of them is 17+31=48.

We notice that all the above referred numbers except of the number 1 (2, 3, 4, 8, 9, 10, 13, 14, 17, 27, 31, 39, 44 and 48) occur as the stars numbers of various constellations (see Table 1). Especially, we note that the total number of stars of Aquarius and Water constellation is 48 divided in 17 and 31 stars respectively, by Eratosthenes.

Another 'specific' triangle is the well known *Pythagorean triangle* which is an orthogonal one with the vertical sides equal to the numbers 3 and 4, while the hypotenuse is equal to the number 5. The perimeter of this triangle is equal to 12 and its area is equal to 6. According to Plutarchus (De Iside et Osiride, Stephanus p. 373, sect. F, lin. 5): '*And the Pythagorean triangle has a height of 3 units of length and its base equal to 4 units of length and its hypotenuse equal to 5 units of length so that the square of the hypotenuse equals the sum of the squares of the two vertical sides. We must therefore assume that the height symbolizes the man, the base symbolizes the woman, and the hypotenuse symbolizes the child of them, and consider Osiris as the beginning, Isis as a reception and Oros as a result. The 'three' is the first odd and perfect number, the 'four' is the square that comes from an even number, the 'two', while the 'five' looks like with the mother and with the father too, because it consists of 'two' and 'three*'. While Plato (Respublica, Stephanus p. 546, sect. c, lin. 1-2) adds: '*The numbers 3 and 4 combined according to the Pythagorean method with the number 5 ((4/3) × (5/4) × (6/5)) give rise to a chromatic octave, which consists of a tetrachord (4/3), a major third (5/4) and a minor third (6/5)*' and Proclus (Platonis rem publicam commentari vol. 2, p. 42, lin. 15-16) notes: '*The numbers of the lengths of the sides are 3, 4, 5, the sum of them or the perimeter of the triangle is 12 and the area of the triangle is 6*'. Also, Proclus (Platonis rem publicam commentari vol. 2, p. 43, lin. 42-43) mentions that: '*Before that, there is no other orthogonal triangle with integers numbers as the side lengths*'.

However, in this decoration of the sky ('cosmos'), the polar constellations are described by the groups of stars, which are analysed in the combinations of the number 3, 4, 7 and 12 (see session 5). These numbers *remind us*, the *Pythagorean Triangle.* The perimeter of this triangle is equal to 12 and its area is equal to 6. If we sum these two 'specific' numbers we found the number 18 (12+6). Also, the sum 3+4=7 as well the multiple 3×4=12 are expressed the two 'marriage' between the 'vertical sides' of this triangle. Indeed, Anatolius ('O*n the Ten and the numbers*

therein', page 12, line 1) reports that the *number 7* is considered as a *holy number* by Pythagoreans, for the reason that it occurs from the 'fertile' marriage 3+4=7, that is *the sum of the 2 vertical sides* of the Pythagorean triangle. Many entities are related with the number 7, i.e. the 7 planets, the 7 lunar phases (Waxing and waning crescent, first and last quarter, waxing and waning gibbous, full moon), the 7 vowels of the Greek language, the 7 strings of Lyre etc. The double of this holy number is 14 ('marriage' 2×7). Also, the double of the other 'holy' number, 12, is 24 (2×12). We note that 14+10=24, which 10=1+2+3+4 ('Holy Tetractys'), while the sum 24+10=34.

Also, the sum ('marriage') of the other two sizes of the Pythagorean Triangle is:
3+5= 8 (or 2^3) and 4+5= 9 (or 3^2)

Moreover, the multiplication ('marriage') of the other two sizes of the Pythagorean Triangle is: 3 × 5= 15 and 4 × 5 = 20

However, the sum of the 'holy number' 7 and the result of the multiplication-'marriage' 12, is equal to 19 (7+12). With the analog way, the sum between the 'holy number' 7 and the multiplication-'marriage' effect of the other two sizes of the Pythagorean Triangle is equal to 22 (=7+15) and 27 (=7+20) respectively.

We notice that all the above referred numbers (3, 4, 7, 8, 9, 12, 14, 15, 18, 19, 20, 22, 24, 27 and 34) occur as the stars numbers of various constellations (see Table 1). Specifically, the number 22, which is the stars number of Sagittarius constellation, was analyzed by Eratosthenes *exactly* as the sum of the numbers 15 and 7. Moreover, the stars number of Centaurus and Lupus, which are presented together by Eratosthenes, is 34, which analyzed by him *exactly* in 24 and 10 stars.

We conclude that all the stars numbers of the constellations according to Eratosthenes (see Table 1), as well as the total number of stars (720=6!) of the constellations and their distribution in five (5) routes on the sky; they seem to have *some numerical relationship* with some important Pythagorean numbers related with the Holy and Greatest Tetractys and the Pythagorean Triangle. Based on *this hypothesis*, we can 'understand and justify' the absence of the numbers 1, 5, 6, 11 and 16, from the sequence of the stars number until the number 20, as well as the presence of some *selected* numbers (22, 24, 27, 31, 34, 39, 44, 48) upper than 20, in this stars number list. The *cyclic numbers* 5 and 6 have additional important interpretations which may have complicated their presence in this list of star numbers. Of course, the number 1 is exclusive because it is impossible to describe with it, a set of stars. The numbers 11 and 16 are not produced from any combination according to previous analysis. In the contrary, the only produced numbers upper than 20, from the above analysis are exactly these selected numbers.

It is important to remind the reader that Eudoxus of Cnidus (390-337 B.C.) studied in the school of Pythagoras and his teacher was Archytas from Tarentum. He met and developed close relations with Plato. Also, Eratosthenes in his work named '*Platonikos*' examines the mathematical foundation of Plato's philosophies (he was taught Mathematics by Theodorus of Cyrene which belonged to the philosophical school of Pythagoras), which means that he has also the knowledge of the Pythagorean theories.

6.2 THE STARS' NUMBERS AND THE PYTHAGOREAN MUSIC HARMONY

It is known that by using the numbers 3, 4, 5, 6 and 12 (three sides, area and perimeter of the Pythagorean triangle), Pythagoras was the first to describe the basic *musical intervals of the chromatic genus* in a gamut of two diapasons, i.e.: 4/3 diatessaron, 5/4 major chromatic third, 6/5 minor chromatic third, 6/3 diapason and 12/3=4=2^2 double diapason or disdiapason.

Because the polar constellations are described by the groups of stars, which are connected with the Pythagorean triangle, we examined the ratios of the star numbers of the polar constellations to one another, in relation to the Pythagorean music harmony and, as a result, we got musical intervals that do belong to the musical chromatic genus (Table 5), with an approximation of the first decimal number.

We remind to the reader that the major or the epogdoon tone (9/8) is the Pythagorean sum of leimma and apotome:
(9/8) = (256/243) x (2187/2048)
Where:

The minor diatonic semitone or leimma is (256/243) or (2^8 /3^5)

The major semitone or apotome of a major tone is (2187/2048) or (3^7 / 2^{11})

Additionally, there is the interval of the Pythagorean comma (531441/524288), namely the difference between the leimma and the apotome of the major tone:
Comma = apotome/ leimma

531441 / 524288 = (2187/2048) / (256/243) or (3^{12}) / (2^{19}) = (3^7 / 2^{11}) / (2^8 /3^5)

Also, if we subtract an epogdoon tone (9/8) from a tetrachord (4/3), namely from two epogdoon tones and a leimma, we will get an epogdoon tone and a leimma, namely a Pythagorean trisemitone.

$$\frac{\frac{4}{3}}{\frac{9}{8}} = \frac{32}{27}$$

Table 5. Ratios of stars of polar constellations and musical chromatic intervals

Cepheus/Ursa Minor and Hercules/Lyra
$\left(\frac{19}{8}\right)=2.375\sim\left[\left(\frac{2}{1}\right)\times\left(\frac{32}{27}\right)\right]\cong\left(\frac{4}{3}\right)^3=2.370$
(Diapason and Pythagorean trisemitone)
Cepheus/Cygnus
$\left(\frac{19}{14}\right)=1.357\sim\left[\left(\frac{4}{3}\right)\times\left(\frac{64}{63}\right)\right]\cong\left(\frac{4}{3}\right)=1.354$
(Diatessaron and minor comma ≈ Diatessaron)
Cygnus/Lyra
$\left(\frac{14}{8}\right)=1.75\sim\left(\frac{4}{3}\right)^2=1.777$
(Disdiatessaron)
Hercules/Draco
$\left(\frac{19}{15}\right)=1.267\sim\left[\left(\frac{4}{3}\right)/\left(\frac{256}{243}\right)\right]=1.266$
(Diatessaron diminished by a leimma)
Draco/Ursa Minor
$\left(\frac{15}{8}\right)=1.875\sim\left(\frac{4}{3}\right)^2\times\left(\frac{256}{243}\right)=1.872$
(Disdiatessaron augmented by a leimma)

We note that by dividing the numerical relations of two musical intervals with the first one of them being bigger than the second, we calculate their 'Pythagorean difference'.

These musical intervals (Table 5) between the Polar constellations are shown in Fig. 10, as arcs into the polar cycle. Following the natural sequence of constellations in the celestial northern pole due to the phenomenon of the precession, and starting from Cepheus constellation, these arcs are approximately (4/3), $(4/3)^2$ and $(4/3)^3$ respectively. We also note that the diametrically occurring polar constellations (Fig. 10) correspond to the same musical note (see in the next session).

The sum of these arcs is equal to twelve tetrachords or, equivalently, to 4.98 (~5) octaves.

$$\left(\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)^2\left(\frac{4}{3}\right)^3\right)^2=\left(\frac{4}{3}\right)^{12}=\left(\frac{2}{1}\right)^{4.98}$$

We must have in mind that Plato in *Timaeus* (The generation of the Soul of the World (35a1-36b6)) contracts a musical scale which covers an interval of four diapason plus a fifth, plus a tone i.e. about 4.75 (~5) octaves.

$$\left(\frac{2}{1}\right)^4\left(\frac{3}{2}\right)\left(\frac{9}{8}\right)=\left(\frac{2}{1}\right)^{4.75}$$

We notice that the aforementioned two musical intervals are almost equal.

Moreover, the sum of the stars of the constellations that have an equinox or solstice position for the three time periods that correspond to the position of Draco and Ursa Minor at the North Pole is 100, 104 and 78 respectively (see session 4). If we take the ratios between these numbers, we get also terms of music harmony, with an approximation of the first decimal number. More specifically:

104/100=1.0400~1.0334=$\sqrt{\frac{2187}{2048}}$ (half an apotome)

100/78=1.2820≅ $\frac{4}{3}$ (diatessaron)

104/78=1.3333=4/3 (diatessaron)

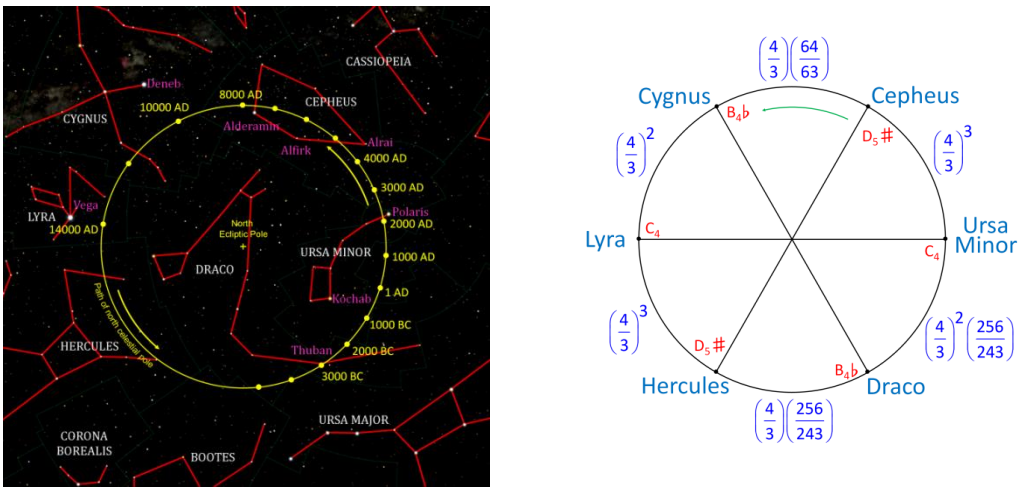

Figure 10: a) The polar constellations during a precession cycle. b) The musical intervals expressed as arcs between the polar constellations. The small musical intervals $\left(\frac{256}{243}\right)$ *and* $\left(\frac{64}{63}\right)$ *in relation to the other ones can be ignored as negligible. The musical notes which are corresponded the each one of these constellations' star number is also marked in the Figure.*

If we sum up the number of stars forming the constellations of each one of the 5 routes in the sky, we get the numbers 113, 123, 157, 133 and 194 respectively (session 3). The analysis of these numbers of stars results in the following ratios that also correspond to *six musical intervals* (Table 6) with an approximation of the first decimal number.

We also notice the following:

a) The 'Pythagorean sum' (see in Appendix) of the musical intervals between the consecutive groups 1, 2 and 2, 3 and 3, 4 and 4, 1 is approximately equal to a diapason (octave):

$$\sqrt{\frac{32}{27}}\cdot\frac{5}{4}\cdot\sqrt{\frac{3}{2}}\cdot\sqrt{\frac{3}{2}}=\frac{5}{2}\cdot\sqrt{\frac{2}{3}}=2{,}041241\cong 2=\frac{2}{1}$$

While the 'Pythagorean sum' of the musical intervals between the groups 4, 1 and 1, 2 is equal to a tetrachord (sesquitertium or diatessaron or 4/3):

$$\sqrt{\frac{3}{2}}\cdot\sqrt{\frac{32}{27}}=\sqrt{\frac{3}{2}}\cdot\frac{4}{3}\cdot\sqrt{\frac{2}{3}}=\frac{4}{3}$$

And the 'Pythagorean sum' of the musical intervals between the groups 2, 3 and 3, 4 is approximately equal to a diapente (sesquialterum or 3/2):

$$\frac{5}{4}\cdot\sqrt{\frac{3}{2}}=1{,}530931\cong 1{,}5=\frac{3}{2}$$

Table 6: The ratio of the sum of the stars of the five celestial routes related with musical intervals

1st and 2nd group: 123/113=1.0885≅1.0886= $\sqrt{\frac{32}{27}}=\frac{21}{20}\cdot\frac{28}{27}$

Half a Pythagorean semiditone $\sqrt{\frac{32}{27}}=\frac{21}{20}\cdot\frac{28}{27}$ *It is equal to a semitone augmented by an enharmonic diesis*

2nd and 4th group: 133/123=1.0813≅1.0886= $\sqrt{\frac{32}{27}}=\frac{21}{20}\cdot\frac{28}{27}$

2nd and 3rd group: 157/123=1.2764≅1.25=5/4
The relation 5/4 express the interval of the major chromatic third.

2nd and 5th group: 194/123=1.5772≅1.5=3/2
The relation (2+1)/2 express the diapente interval.

4th and 5th group: 194/133=1.4586≅1.5=3/2

1st and 4th group: 133/113=1.177~1.2247=√(3/2)
It *express the half of the diapente interval*

3rd and 4th group: 157/133=1.1805≅1.2247=√(3/2)

3rd and 5th group: 194/157=1.2357≅1.2247=√(3/2)

1st and 5th group: 194/113=1.7168~1.7321= $\sqrt{3}$

It express the half of the interval diapason plus diapente, i.e. $\sqrt{\frac{2}{1}}\cdot\sqrt{\frac{3}{2}}=\sqrt{3}$

1st and 3rd group: 157/113=1.3894≅1.4142=$\sqrt{2}$

It express half of the diapason (octave) $\sqrt{\frac{2}{1}} = \sqrt{2}$*, which is the most dissonant musical interval, also called Diabolus in Musica.*

It is clear that the saying of the Pythagorean Philolaus, (Nicomachus, *Manuale Harmonicum*, 9, 1, 14-15) "*the size of a harmony (=octave) is equal to the sum of a diatessaron and a diapente*", is applied among these three 'Pythagorean intervals', (2/1)= (4/3) x (3/2)).

We should note that these four groups correspond to the constellations that are commencing from the North Celestial Pole ending up south, on the Zodiac cycle.

b) The musical intervals between the 5th group and the 1st, 2nd, 3rd and 4th group, respectively, are equal to $\sqrt{3}$, $\frac{3}{2}$, $\sqrt{\frac{3}{2}}$ and $\frac{3}{2}$. However, all of these musical intervals are related to the diapente interval (3/2), since they are respectively equal to half of a diapente augmented by a diapason (octave), a diapente, half of a diapente and a diapente.

Let us not forget that the stars of the 5th group belong to constellations that are extending to two different directions beneath the Zodiac, to the west and to the east, with the brightest star Sirius.

We conclude that the ratios of the constellations' stars of the first four groups that correspond to constellations extending from the north pole to the zodiac cycle are connected to the *octave* and the ratios of the constellations' stars of the 5th group, which concern constellations that are spread out to the east and the west of Sirius in relation to the stars of the previous four groups, are associated with the *diapente interval (3/2).*

So, we notice that the ratios of the number of stars of the particular categories of constellations are not random, but express fundamental musical intervals. It should be noted that Eudoxus of Cnidus, who defined these constellations, besides his knowledge of Pythagorean philosophy that he possessed due to his studies, developed, among other things, the *theory of proportions*, which bridged the gap between the symmetrical and asymmetric mathematical concepts. In the above analysis, we notice that the *application of the analogies* between the stars of various constellation groups leads to Pythagorean music harmony terms.

6.3 PYTHAGOREAN NOTES RELATED TO THE NUMBER OF STARS

Based on the previous conclusion, we studied the numbers of the stars that form constellations in relation to the Pythagorean theoretical and applied music from the perspective of modern musical acoustics. The constellations according to Eratosthenes are described using 23 recurrent numbers of stars: 2, 3, 4, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 19, 20, 22, 24, 27, 31, 34, 39, 44 and 48. We considered that these numbers represent the harmonics of the *composite sound* produced by the rotation of the *eighth sphere* of the Pythagorean theory of the Harmony of the Spheres. We assumed that this composite sound has a *fundamental frequency or first harmonic* that is determined by a hypothetical constellation of one star, based on which the musical intervals of all the other 23 harmonics will be calculated. Thus, the 23 numbers of stars, become 24, as the constellation of reference was added to them. We note that Alexander mentions the following 'specific meaning' regarding the number 24: "*because the world is equal to 24 elements, since there are 12 zodiacs, eight celestial spheres and four Platonic elements*" (*Ibidem,* 835, 16-18).

As it is possible to create a harmonic series on the basis of any fundamental note, in our case, we will regard the C1 note as the fundamental note, according to the European equal temperament in order to simplify our calculations as we match each harmonic of the series to a particular note of the European musical scale. C1 is the first C on the piano's clavier, known as note 'do', with a frequency of 32.7 Hz. The fundamental or first harmonic determines the musical pitch of the sound.

The second harmonic creates the interval of an octave with the fundamental harmonic $\frac{f_2}{f_1} = 2 = 1 \cdot 2 = \left(\frac{1}{1}\right) \cdot \left(\frac{2}{1}\right)$ and, consequently, represents the same note, yet an octave higher. The third harmonic creates the interval of $\frac{\boldsymbol{f}_3}{\boldsymbol{f}_1} = 3 = \left(\frac{2}{1}\right) \cdot \left(\frac{3}{2}\right)$ with the fundamental one, namely an octave and a perfect fifth or a twelfth. It represents the G2 note, which is different from the one of the fundamental harmonic.

The fourth harmonic creates the interval of $\frac{\boldsymbol{f}_4}{\boldsymbol{f}_1} = 4 = \left(\frac{2}{1}\right)^2$ with the fundamental one, namely a double octave, and represents the same note with the fundamental harmonic, yet two octaves higher (C3).

And so on… This analysis shows that every frequency of the harmonic series expresses a sinusoidal

sound which creates either a euphonic or a dissonant musical interval in relation to the fundamental frequency. Thus, the notes of the scale of the European equal temperament are matched to the constellations in an absolutely objective and univocal manner. A more detailed analysis of all the harmonic terms that resulted from this assumption is provided in our manuscript entitled "*The Catasterismi of Eratosthenes and the Pythagorean Harmonic*", in Proceedings of the International Conference "Ancient Greece and the Modern World" (Ancient Olympia, Greece, August 28-31, 2016), p. 588-600, Editor S. Paipetis, Libanis Pub., Athens, Greece, 2017.

The results of the previous analysis are shown in Table 7. We note that these notes create the ambitus C2-G6, namely four octaves (C2-C6) and a perfect fifth interval (C6-G6). This ambitus is only lacking a single tone from the one mentioned by Plato in the *Creation of the World Soul (Timaeus,* 35a1-36b6). In this text Plato with a mathematical process based on his Tetractys, he found a musical scale that has exceeds four octaves, a fifth interval and one tone. The above analysis has exceeds four octaves and a fifth interval. The small difference of one tone indicates coincidence results of two procedures.

Also, these notes reduced into an octave and approximating notes that are either natural or altered by a sine, provide an octave system (=minor sixth) from two conjuncted tetrachords (the notes C, C$, D, D$, E, F and G, A, B@ in ascending order), which could be played by a flute, for example. Moreover, the notes C, D, E, F, G, A, B@ sound like tuning a seven-string lyre (via two conjuncted ascending Lydian tetrachords, proving Heraclitus Ponticus right (Homeric Problems, 12)).

Moreover, a tetrad of notes creating the 'cycle of the fifths' can be identified in this ambitus, which means that the respective numbers of their stars are integer multiples of the divine musical ratio (3/2), according to Aratus (*Διάφορα περί ζωδιακού - On the zodiac*, 531, 13-15). These notes are: C4 (8), G4 (12=8 × (3/2)), D5 (18=12 × (3/2)), A5 (27=18 × (3/2)).

Furthermore, we remind the polar constellations of Fig. 10 and we assign the proper note to each of them according to Table 7. The sequence of the six polar constellations provides us with a sequence of notes. Actually, starting from the note D5$ (Cepheus) the melody moves downwards an interval of a tetrachord and reaches the note B4@ (Cygnus). Then it continues its movement of two tetrachords and reaches the note C4 (Lyra). Finally it moves an interval of three tetrachords and reaches the note D5$ (Hercules). From Hercules to Cepheus the melody repeats exactly itself completing the musical motive (D5$, B4@, C4, D5$, B4@, C4, D5$) i.e. a 'melody' which repeats itself every ~26,000 years.

We conclude that all of the notes of Table 7, which are the result of the previous musical analysis and correspond to the number of stars of the constellations, could be created a "*partiture of the sky*".

Table 7: The notes corresponding to the harmonics of the composite sound of the 8th sphere (sphere of fixed objects) with C1, as note of reference

Serial No.	Numerical relations	Stars' number or harmonic order	Corresponding note with C1 as note of reference	Constellations
1	1 × 1	1	C1	
2	2 × 1	2	C2	Serpens
3	3 × 1 (3/2) × 2	3	G2	Deltoton/Triangulum, Canis Minor
4	2 × 2 (4/3) × 3	4	C3	Aquila, Sagitta, Ara
5	1 × 7	7	slightly diminished B3@ note	Pleiades, Lepus, Coma Berenices, Corvus
6	2 × 4 (4/3) × 6	8	C4	Ursa Minor, Auriga, Lyra
7	3 × 3 (3/2) × 6	9	D4	Delphinus, Corona Borealis
8	2 × 5	10	E4	Crater, Lupus
9	3 × 4 (3/2) × 8 (4/3) × 9	12	G4	Piscis Austrinus
10	1 × 13	13	significantly augmented discordant A4@ note	Cetus, Eridanus
11	2 × 7	14	slightly diminished B4@ note	Bootes, Cygnus
12	3 × 5 (3/2) × 10	15	~B4@	Cassiopeia, Drago
13	1 × 17	17	C5$	Aquarius, Ophiuchus, Aries, Orion
14	3 × 6 (3/2) × 12	18	D5	Pegasus, Cancer, Taurus
15	1 × 19	19	D5$	Cepheus, Heracles, Leo, Scorpius, Perseus, Gemini

16	4 × 5 (4/3) × 15	20	~ E5	Virgo, Canis Major (Sirius), Andromeda
17	2 × 11	22	significantly diminished discordant F5$ note	Sagittarius
18	4 × 6 (3/2) × 16 (4/3) × 18	24	G5	Ursa Major, Capricornus, Centauri
19	3 × 9 (3/2) × 18	27	~A5	Argo, Hydra
20	(3/2) × 2 × 10 × (31/30)	31	B5 augmented by approximately half a semitone	'*Hydria*' (it means 'jug')
21	2 × 17	34	C6$	Centauri and Lupus*
22	38 × (39/38) (3/2) × 26	39	D6 augmented by approximately three quarters of a semitone	Pisces
23	4 × 11 (4/3) × 33	44	significantly diminished discordant F6$ note	Hydra and Crater and Corvus*
24	4 × 12 (3/2) × 32 (4/3) × 36	48	G6	Aquarius and '*Hydria*'*

* referred to as a single constellation

9. CONCLUSIONS

In this work, we study the distribution of the constellations in the sky and the distribution of the star numbers in these constellations based on the 'Catasterismi of Eratosthenes'. We also discuss these star numbers in comparison with some well-known Pythagorean numbers and the music harmony, as this is expressed through numerical analogies of known Pythagorean numbers. We note that the celestial map that was preserved through the works of Aratus and Eratosthenes based on the work of Eudoxus of Cnidus which has taken the knowledge of the Pythagorean theories.

In the *'harmony of the spheres'*, Pythagoreans combine Astronomy with Music claiming that in a harmonic and spherical universe everything is governed by simple laws that can be expressed through the numbers of the *'Holy Tetractys'*. Tetractys is geometrical expressed with an isosceles triangle. However, the *number 4*, which corresponds to the first four natural numbers included in Tetractys, is symbolized in ancient time with the G*reek letter 'delta (Δ)'*, the symbol of which is similar to an isosceles triangle. It should be noted that there is a constellation, called "*Deltoton*", which means the Greek '*letter Δ (delta)*' and is presented *exactly* as an isosceles triangle. Today this constellation is named '*Triangulum*.
Both Aratus and Eratosthenes referred to the same constellations and presented them with the *same hierarchical order*. We mention that these constellations have been followed five (5) routes on the celestial sphere. The number 5 is a 'specific' number for the Pythagoreans. Among others, it is the sum of the squares of the first two numbers (1 and 2): $5=4+1=2^2+1^2$. It is remarkable that this interpretation of the *number 5* is appeared in the *routes of the constellations*. Specifically, the four (2^2) description routes of constellations are commencing from the North Celestial Pole, ending South, on the Zodiac and the fifth route (1^2) extends in two different directions beneath the Zodiac, towards west and east, with the brightest star Sirius (or the constellation of Canis Major) at a center.

Eratosthenes reports the exact number of stars in each constellation that he describes. If we sum up all these stars, we get the sum of 720 which is equal to 6! (= 1 × 2 × 3 × 4 × 5 × 6). This number of stars is divides to 156 polar and their related constellations' stars, 282 zodiac's constellations stars and 282 other constellations' stars. The number 6 is another 'specific' number for the Pythagoreans express '*the Soul of Cosmos*'. Plutarch considered the number 36 (6^2) that is the sum of eight first natural numbers, as the '*Rotating Cosmos*'. The Greek word '*cosmos*' is produced by the verb '*κοσμώ (cosmo)*' which means '*decorate*'. The total number of stars that *decorate* the starry sky (heavenly sphere) according to Eratosthenes is related with the number 36 (720=2 × 10 × 36), which described the 'Cosmos', when the ideal shape of 'Cosmos' is the sphere, according to the Pythagoreans. In the contrary, the description of the constellations includes 918=17 × 54 stars, according to Ptolemy, when the number 54 is equal to the sum of the terms of the Plato's '*Greatest Tetraktys*' expressed the '*three-dimensional Cosmos*'.

We examined the ratios of the number of stars forming the constellations of each one of the 5 routes in the sky (113, 123, 157, 133 and 194 stars) to one another, in relation to the Pythagorean music harmony and, this analysis results in the ratios that correspond to *six musical intervals.* We also note that, the ratios of the constellations' stars of the first four groups that correspond to constellations extending from the north pole to the zodiac cycle are connected

to the *octave* and the ratios of the constellations' stars of the 5th group, which concern constellations that are spread out to the east and the west of Sirius in relation to the stars of the previous four groups, are associated with the *diapente interval (3/2).*

In this decoration of the sky, the polar constellations are described by the groups of stars, which are analysed in the combinations of the numbers which they remind us, the *Pythagorean Triangle.* It is known that by using the numbers 3, 4, 5, 6 and 12 (three sides, area and perimeter of the Pythagorean triangle), Pythagoras was the first to describe the basic *musical intervals of the chromatic genus* in a gamut of two diapasons. We examined the ratios of these star numbers of the polar constellations to one another, in relation to the Pythagorean music harmony and, as a result, we got musical intervals that do belong to the *musical chromatic genus*.

Moreover, the stars numbers of the constellations have *a kind of numerical relationship* with some important Pythagorean numbers related with the Holy Tetractys and Plato's Greatest Tetractys and the Pythagorean Triangle. They classified in groups of number 2 and its exponents (2, 2^2, 2^3), of number 3 and its exponents (3, 3^2, 3^3), of multiples of the number 5 and number 11, of number 7 and number 17 and the double them, of the number 12 and its multiples, of the number 13 and its triple, of the numbers 18 and number 19.

Also, these 23 recurrent numbers of stars which are described the constellations represent the harmonics of the *composite sound* produced by a *fundamental frequency or first harmonic* that is determined by a hypothetical constellation of one star. The most of them are near to the notes of the scale of the European equal temperament. In other words, all these notes, which correspond to the number of stars of the constellations, could be created a "*partiture of the sky*".

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