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PYTHAGORAS' MATHEMATICS IN ARCHITECTURE AND HIS INFLUENCE ON GREAT CULTURAL WORKS

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ABSTRACT

Pythagoras' life, teaching and contribution in science and philosophy has been transfigured by legend, which hardly can be separated. Tracing his fingerprints of mathematical nature is attempted here, based on evidence from great technical works and temples accomplished during his time in Samos and Magna Graecia. The application of the Pythagorean triples in the design of the Athena temple at Paestum built in c. 520 BC has already been established and was considered to attest the Pythagorean consciousness of the architect. Similar conclusions are also drawn in this article from the layout of the Polycratean temple of Heraion in Samos, where the earliest application of Pythagorean mathematics and proportions is disclosed in this article. It is also demonstrated that the achieved accuracy in pre-positioning of the Eupalinos' tunnel mouths and the well-designed maneuver at the crossing indicate the involvement of a mathematical mind supporting the engineering skills of Eupalinos. By comparison with the Hellenistic temple of Apollo at Didyma, where the systematic application of the Pythagorean triples is again revealed in temple modeling and layout, it is concluded that the geometrical method of design of the ancient temples and the concept of harmonic proportions was fully developed in Pythagoras' time and his philosophy of proportions in architecture, amalgamated later with Plato's ideas, prevailed since then until the present.

Keywords: Pythagoras; Pythagorean triples; harmonic proportions; Heraion of Samos; Eupalinos' aqueduct; Paestum, Magna Graecia; Didyma.

1. INTRODUCTION

Samos in its heyday hosted men of letters, outstanding architects, artists, and scientists. As conjectured, there must have been considerable geometrical activity in the sixth-century BC in Samos, required by the immense building projects. It is also believed that the origins of Greek mathematics lie in Greek engineering and that the building projects had greater influence on Pythagoras than Pythagoras had on the building works. Not only the Eupalinos' aqueduct, but primarily the temple of Heraion, the so-called Samian "labyrinth", presupposed careful application of mathematics and project design (Rihll and Tucker, 2003: 416).

Geometry is generally held to have been applied first in Babylonia and Egypt. It owed its development in Egypt to the practice of land measurement because the overflow of the Nile would disorder the boundaries of land pieces. It was Thales, who after a visit to Egypt first brought the study of geometry to Greece. Not only did he make numerous discoveries himself but laid the foundations for many other discoveries on the part of his successors.

Thales was regarded as the patron saint of mathematics even in the fifth century (Burkert, 1972: 413). Pythagoras has grown in this intellectual atmosphere. He was born in Samos in c. 570 BC and left for Italy most likely in 532/531 BC because of the oppressive tyranny of Polycrates (Bunkert, 1972: 110). He was in Samos when the works for the Heraion were in progress and of course during the completion of the tunnel for which the works started as early as c. 550 BC (Kienast, 2005: 37).

As eloquently epitomized by Burkert, there is no doubt of the historical reality of the Pythagorean society and its political activity in Croton; but the Master himself can be discerned, primarily, not by the clear light of history but in the misty twilight between religious veneration and the distorting light of hostile polemic. Pythagoras and the Pythagoras' legend cannot be separated (Burkert, 1972: 120).

It is hoped at least that the Master's fingerprints can be traced in great technical works of his time. I believe therefore that it is worth tracing and studying any indication of mathematical nature in the architectural and technical masterpieces which were being realized during Pythagoras' time in Samos, and Magna Graecia. This is partly the aim of this paper, along with the investigation of the origin of mathematical knowledge applied.

Our objectives, the steps we follow, start of course from Samos, and extend in Croton's metropolis in Achaea, near Aegion, and finally in Magna Graecia. The monuments examined, temples and the Eupalinos' aqueduct, fall in the span of Pythagoras' life. A further step brings us to the Hellenistic temple at Didyma in the domain of Miletus, Thales' place of origin, for the study of the evolution of Pythagoran ideas in the next centuries.

A polemic against Pythagoras extends from antiquity to the present and some scholars consider today that Pythagoras was not "a master geometer, who provides rigorous proofs, but rather someone who recognizes and celebrates certain geometrical relationships as of high importance" or even that "the traditional stories of discoveries made by Thales or Pythagoras must be discarded as totally unhistorical". Therefore, our study of the Pythagorean triples[1] extends into the Babylonian mathematics.

It is revealed for the first time that the layout of the temples at Heraion in Samos, Trapeza near Aegion, and Apollo temple at Didyma is designed based on Pythagorean triples; the method of temples' design and generation of the triples are also elucidated. Alternative methods of Pythagorean triples generation are investigated for the temples examined. The ingenuity of the Old Babylonian mathematics is appreciated, but it is concluded that Neugebauer's persistence on the use of generating functions is an unnecessary anachronism.

2. GEOMETRIC DESIGN OF THE LAYOUT OF LATE ARCHAIC TEMPLES BASED ON PYTHAGOREAN TRIPLES

Layout surveying for important constructions in ancient Egypt was both an important procedure and ceremony as described by Paulson (2005). At the beginning of the construction of the pyramid, the priests, builders, and perhaps the pharaoh himself would have performed a "stretching of the cord" ceremony. The Egyptian phrase for a surveyor was a "rope stretcher" and surveying was known as "stretching a rope". In fact, a calibrated rope was one of the tools used in surveying. Several tombs from the New Kingdom era about 1100 BC show the tomb owner overseeing men using ropes to measure fields, presumably to calculate the taxes for yield of these fields (Paulson, 2005: 2/12).

In the following the dimensions and layout of Archaic temples in Samos and Magna Graecia are investigated for possible Pythagoras' influence; the examined monuments were built in the second half of the

[1] A Pythagorean triple consists of three positive integers a, b, and c satisfying the Pythagorean theorem, such that $a^2+b^2=c^2$

6th century BC, a period of Pythagoras's presence there.

2.1 The second dipteros temple of Heraion in Samos

The second dipteros temple of Heraion in Samos is described as a labyrinth (Pliny, Natural History 34.8 3). The enigmatic term "labyrinth" must be a popular name of the gigantic temple of Hera in her sanctuary in Samos, which through its double and triple rows of over a hundred columns must have given the impression of labyrinthine complexity (Kyrieleis, 1990: 17). Despite that, the geometric design of the temple can be greatly simplified if the geometric rules applied are understood, a task which is attempted here.

During the tyranny of Polycrates, work began on a new temple, known as the second dipteros (Hellner, 2002: 168) or Polycratean temple, on a stylobate measuring 55.16×108.63 meters (magenta in Fig. 1), even larger than the first dipteros temple. It is revealed that the dimensions of the new temple signal a change of proportions at the Heraion in Samos, a new trend which was spread and applied to other Late Archaic temples soon. Thus, by comparison to the ratio 2:1 of the first dipteros temple, the stylobate's ratio at the second dipteros temple is 108.63: 55.16 = 1.9694 = $(2-\frac{1}{32})$. It is indeed a minor numerical change by itself, associated however with a "latent" significant evolution at the level of geometric design, which is the expert application of Pythagorean triples. As estimated from the temple plan (Gruben and Kienast, 2014: Beilage 5), the columns are about 0.3 to 0.35 meters apart from the outline of the stylobate. Thus, if the stylobate dimensions are reduced by d=0.65 m, the dimensions of rectangle envelope around the outer colonnades, blue in Fig. 1, are calculated as follows:

(55.16-0.65) × (108.63-0.65) = 54.51 m×107.98 m and their ratio:

107.98/54.51 = 1.98092 equals to 208/105 (1.98095).

Therefore, the blue rectangular envelope, which circumscribes tangentially the outer colonnade, is a Pythagorean one corresponding to the Pythagorean triple (105, 208, 233). Furthermore, it is:

$\frac{54.51}{105}$ = 0.519143 m and $\frac{107.98}{208}$ = 0,519135 m

and this implies a length unit at the Heraion temple, a cubit of 0.519 or ~0.52 m, impressively close to the Samian cubit calculated below from the tunnel measurements.

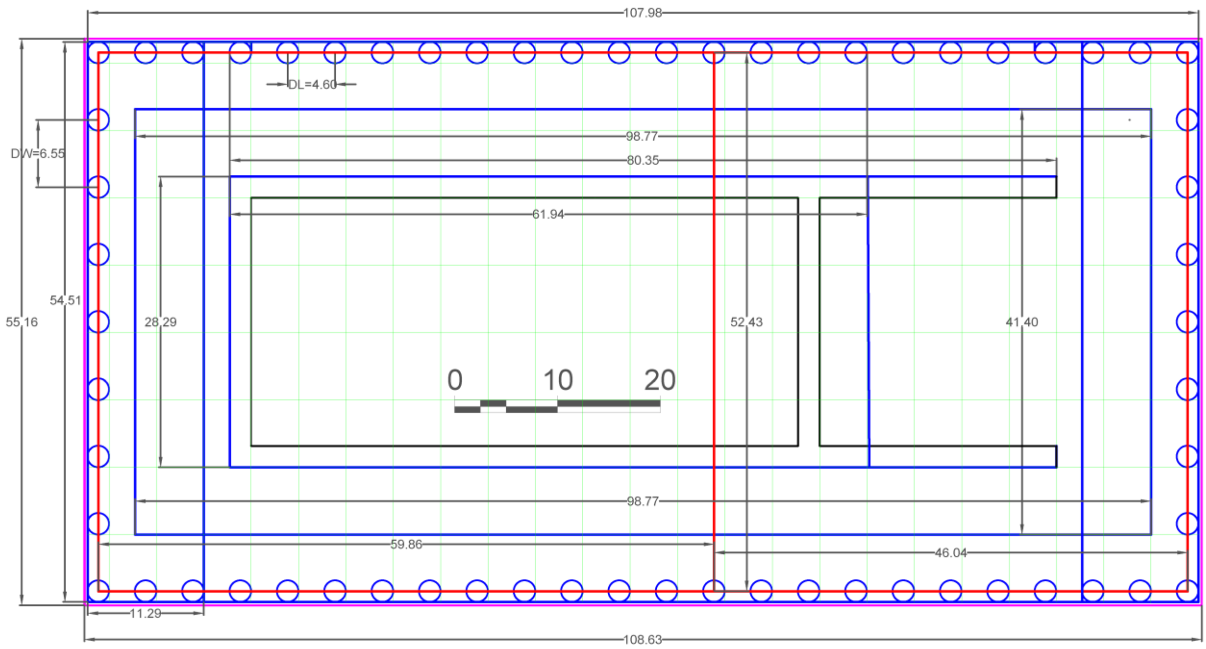

Figure 1: Simplified plan of the second peripteros temple of Heraion in Samos, based on the plan 5 by Gruben and Kienast (2014). The stylobate (magenta), the Pythagorean rectangles in blue (No 1, 2, 3 and 4 of the Table 1), the subdivision of the interaxial rectangle in red (No 5 and 6 in Table 1) and the inferred rectangular grid in green.

The interaxial distances in meters, for a column radius of one cubit at the base, are:

DL= (107.98-2x1.04)/23 = 4.60 m along the length and DW = (54.51-2x1.04)/8 = 6.55 m along the temple width. Their ratio is remarkably close to the square root of 2, which implies that DW equals the diagonal of a square with side DL, and that DW and DW served as modules of a rectangular grid.

Apart from the rectangle of the outer colonnade, some smaller Pythagorean rectangles are also delineated on the temple layout in blue and are summarized in the Table 1. They are characterized as Pythagorean because their sides and diagonals are proportional to a Pythagorean triple shown in the Table 1. The rectangles No 1 and No 2 in the table correspond to the envelope of the outer and inner colonnade respectively (Fig. 1). The rectangle No 3 is repeated twice and surrounds three rows of nine columns each, at both faces of the temple. The inner structure of the temple is based on the Pythagorean rectangle No 4 in the Table 1.

Table 1: Pythagorean rectangles revealed in the Heraion temple

Rectangle, No	Length, m	Width, m	Pythagorean triple
1	107.98	54.51	(105, 208, 233)
2	98.77	41.40	(5, 12, 13)
3	11.29	54.51	(5, 12, 13)
4	61.94	28.30	(115, 252, 277)
5	59.86	52.43	(48, 55, 73)
6	52.43	45.96	(48, 55, 73)

It is amazing that the rectangle formed by the axes of the outer colonnade, red in Fig. 1, measures 52.43x105.00 m and has a sides ratio 2.003, practically 2: 1, the harmonic ratio considered to correspond to the octave (diapason). The interaxial rectangle can be subdivided in two Pythagorean rectangles, No 5 and 6, respectively 13xDL and 10xDL long both proportional to the Pythagorean triple (48, 55, 73).

In short, it is supported that there is strong evidence of thorough mathematical design in the temple layout, including the multiple application of Pythagorean triples and simple proportions which are implemented by the use of a rectangular grid of dimensions DL by DW.

The application of Pythagorean rectangles provides a better design control because in addition to the intended dimensions of the rectangle sides, the diagonal is known in in round length units, so that right angles and the dimensions are more accurately implemented. This is particularly significant for the colonnades but is also locally applied through the rectangles 3 and 4 of the Table 1 and this indicates the great care for geometrical perfection. Besides, the application of the grid, a technique already in use by the Egyptians, facilitates the allocation and control of the architectural plan on the ground.

2.2 The Trapeza temple

Another Archaic temple, the layout of which is based on a Pythagorean rectangle, is the peripteral hecatombedos Doric temple at Trapeza of the city of Rhypes, a city-state of ancient Achaean Metropolis of Croton in Magna Graecia. The temple was founded in the decade 520- 510 BC (Vordos, 2016); however, Kanellopoulos and Kolia (2011: 148) date the temple earlier in 530-525 BC. The temple dimensions in meters, as given by Hellner and Gennatou (2015: 120), and the values of the Length to Width (L/W) ratio are summarized in the Table 2. It is underlined that the euthynteria sides correspond precisely to the Pythagorean triple (8, 15, 17) multiplied by 7, given that 31.56/16.84= 1.8741~15/8= 1.875 and that 1.875/1.8741 =1.0005. It is interesting too that the Trapeza temple is contemporary or older than the temple of Athena at Paestum. The calculated length unit **u** from the euthynteria dimensions is:

u= 31.56 m/15x7 = 16.84 m/8x7 = 0.3006 m.

It is noted that the crepis and stylobate dimensions are also expressed in round numbers in terms of the model unit, shown bold in the Table 2; it should therefore be examined how this unit **u** is correlated to the Attic foot. The location of the temple on the route from Delphi to Italy is also noted and the point is raised whether it could be corelated with Pythagoras' visit to Delphi. In any case, another reasonable way of Pythagoras' influence is through the city of Croton, an Achaean colony in Magna Graecia.

Table 2: Dimensions of Trapeza temple from Hellner - Gennatou (2015) in meters and model units (u)

Level	Length m/**u**	Width m/**u**	L/W
Euthynteria	31.56 **105**	16.84 **56**	15/8 (*1.875*)
Crepis	31.25 **104**	16.45 **54.75**	19/10 (*1.9*)
Stylobate	30.51 **101.5**	15.64 **52**	39/20 (*1.95*)

2.3 The Athena temple at Paestum

Especially important is the envisaged influence of Pythagoras in the design of the temples in Magna Graecia in the second half of the 6th century BC. Confirmation on that comes from the Athena temple at Paestum examined for "Pythagorean qualities" by Nabers and Wiltshire (1980). The temple is commonly dated to around 510 BC and demonstrates the application of Pythagorean triples in southern Italy during Pythagoras' time there, roughly 532/1 to 494/3 BC.

Nabers and Wiltshire (1980), using precise measurements of the temple, independently established, discovered that two Pythagorean triples were used in the design of the temple, one on the plan and a second one on the flank elevation. The Pythagorean triangle present in the plan of the Athena temple at Paestum is a version of the basic or "primitive" Pythagorean triangle (5, 12, 13), enlarged by a factor of 8. Yet another Pythagorean triangle exists in the design of the temple on the flank elevation with the sides (28, 96, 100), which is a version of the primitive Pythagorean triple (7, 24, 25), enlarged by a factor of 4. Therefore, Nabers and Wiltshire (1980: 215) conclude that the teachings of Pythagoras in southern Italy affected the design of the Athena temple and in particular that: "Here we have a structure of fairly certain date, contemporary with Pythagoras himself, which at least attests the Pythagorean consciousness of its architect and may reflect broader philosophical and political conditions at Paestum as well. Finally, as a physical monument, it manifests in an empirical way the fundamental Pythagorean proposition that "things are numbers" and suggests that the cosmic order apparent to the Pythagoreans in the musical scale may also be expressed in architectural form".

The application of a Pythagorean triple also on the elevation is particularly important for the interrelationship of proportions in three dimensions, projected from the plan layout to the whole monument.

2.4 The echo of Pythagorean harmony on the design of the Apollo temple at Didyma

The application of Pythagorean triples in ancient architecture became widespread as documented by Ranieri (1997: 210) who attributed to Pythagoras a rule of triads. For comparison's sake, a short reference to the Hellenistic Apollo temple at Didyma follows, selected as an outstanding case study. The temple is the best preserved and among the largest Greek temples (Weber 2011: 33), it has been studied systematically since long and reflects the Pythagorean-Platonic ideas of harmonic design. It is therefore reviewed here for investigating possible Pythagorean tradition a few centuries after the Heraion and Athena temples.

Birnbaum (2006) performed a thorough harmonic analysis of the dimensions of the Apollo temple at Didyma by calculating ratios of rectangle sides and other dimensions that correspond to musical consonances. Certain ratios in architecture are considered harmonic, by analogy to vibrating strings which sound at musical intervals if their lengths are in simple, rational numerical relationships. So, the ratio 2: 1 is considered to correspond to the octave (diapason), 2: 3 to the fifth (diapente) and 3: 4 to the fourth (diatessaron). It is underlined by Birnbaum (2006: 12) that the connection of numbers with music by the Pythagoreans gave the numbers an over-mathematical meaning and was used as a fundamental insight into the essence of reality, in the belief that the metaphysical order is expressed in the musical harmony. A rectangle is considered harmonic if the sides ratio deviates less than one percent from a musical interval. The crepis outline of Didymaion with a side's ratio of 197/100, is close to the ratio 2: 1 but not enough to be considered as harmonic. By contrast, the hypothetical rectangle which lies in the plan of the temple exactly in the middle between the second and third crepis step is harmonic with sides ratio exactly 2:1 (Birnbaum 2006:94). In short, Birnbaum (2006: 181) concludes that an interpretation of dimensions in connection with the Pythagorean-Platonic theory of numbers is not only possible, but rather mandatory.

It is understood that Birnbaum investigates Didymaion in Povilioniene's sense (2013: 96), as a link between music and architecture, as a philosophical-aesthetic problem of harmonious universality in which interaction between the art of sounds and visual art reveals itself most clearly through a constructive "common denominator" – the use of numbers, proportions and symmetry.

Particularly important and insightful for the plan design of the Didymaion temple is the system of inscribed letters at the upper blocks of the euthynteria, described and ingeniously interpreted by Weber (2011: 33). In places of the temple's euthynteria exist letters, carefully carved like inscriptions, at an average distance of b = 1.324 m, where b stands for the German term "Buchstabenabstand". Weber interpreted these letters as the legend of a grid of 44×88 square cells, green in Fig. 2, with elementary cell dimensions 1.324X1.324 m and total dimensions 58.256 × 116.512 m. On the plan eight large squares can be shaped into two rows, each of four squares of 22x22 cells. The crepis ABEF, shown in red, is larger than the green grid and measures 60.085 x118.340 m. So, the 2:1 ratio of the grid becomes in the crepis outline 197:100 and Weber investigated why this change from the "nice" 2:1 ratio (88:44) to an "ugly" one. More importantly, he also recognized that the regular distance between the letters (b = 1.324m) equals to one quarter of the interaxial distance, taken as the modulus, M, of the temple (M = 5.296 m), and to one half of the square bases of the columns. Incidentally, it is reminded that at the Heraion temple in Samos the sides ratio of the interaxial rectangle of the outer colonnade is 2:1.

Adherence to the harmonic theory prevented the researchers from recognizing that the temple layout originated from an "ugly" Pythagorean rectangle of the crepis outline as a background from which the "nice" rectangle of the grid resulted. The crepis outline is in fact composed of two equal Pythagorean rectangles ABCD and DCEF (Fig. 2) with sides 118.34 m and 60.085/2 = 30.043 m and sides ratio equal to 3.9394. This ratio practically equals to 63/16 = 3.9375 and therefore the rectangle sides 118.34 m 30.043 m are proportional to the members 16 and 63 of the Pythagorean triple (16, 63, 65).

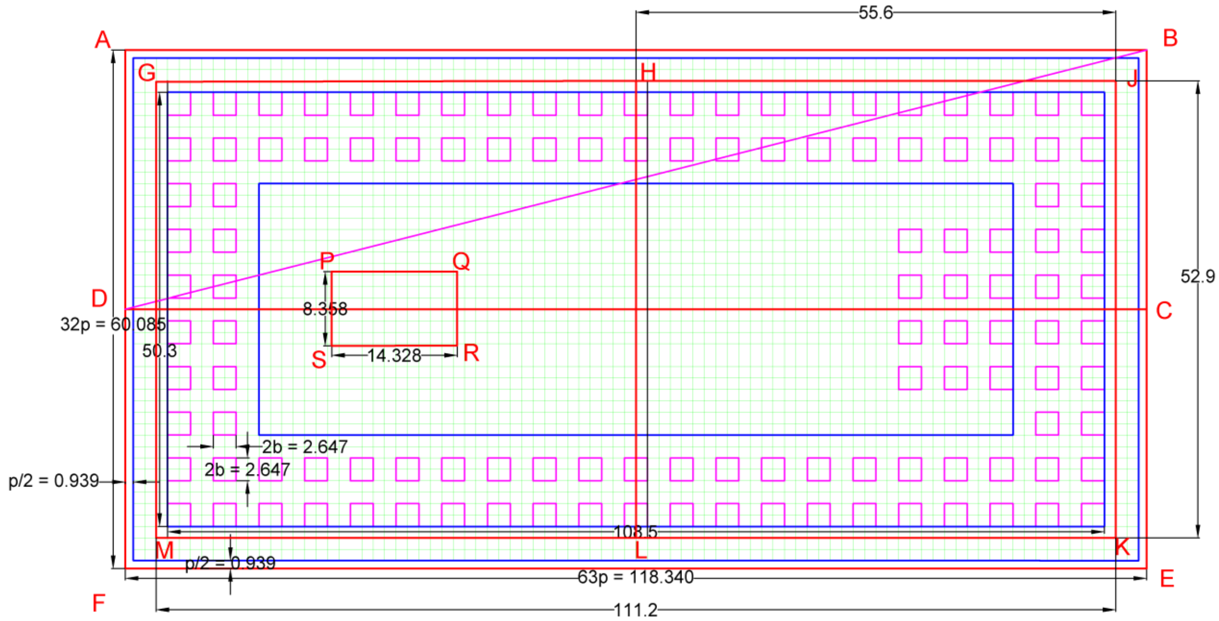

Figure 2: Division of the crepis (ABEF) of the Apollo temple at Didyma into the Pythagorean rectangles ABCD, DCEF, GHLM and HJKL (red), overlying the green grid. The Pythagorean rectangle of the Naiskos, PQRS in red, and the square bases of the columns (magenta) are also shown. Modified from Weber (2011).

The respective rectangles of the Pythagorean model, A′B′C′D′ and D′C′E′F′ (Fig. 3), measure 16x63 dimensionless model units, named here "Pythagorean" units (p). The equivalent of p-unit on the temple equals AB/63= 118.34/63 = 1.878 m. By shifting in the model of Fig. 3 the outline A′B′E′F′ inwards by half a model unit, a "nice" rectangle results 31x62 in size, composed of eight squares 15.5x15.5 (p) units in two rows like the temple.

By analogy, by shifting the crepis outline ABEF of the temple (Fig. 2) by the equivalent of p/2=1.878 m/2 = 0.939 m the nice rectangular of the green grid results.

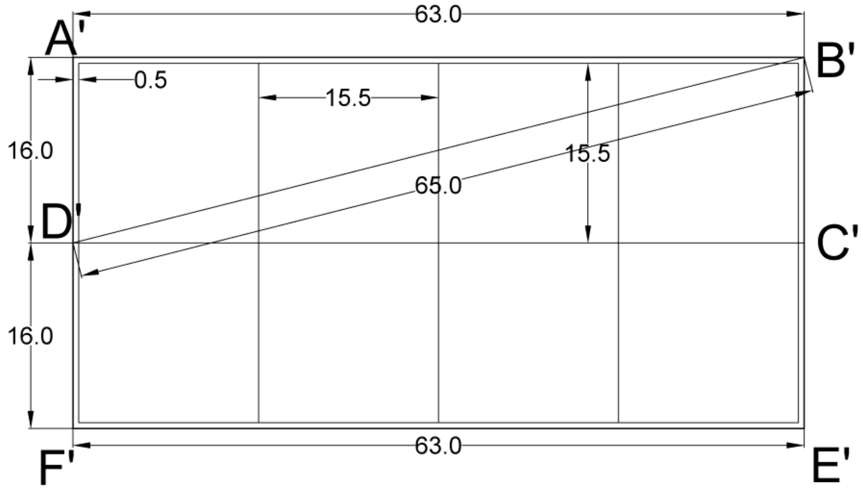

Figure 3: Pythagorean rectangles A′B′C′D′ and D′C′E′F′ proportional to the (16, 63, 65) triple, as a model of the Apollo temple crepis.

This relationship of the Pythagorean model and the actual geometry of the temple provides an insight into the process of architectural design. First, the geometrical pattern is designed on the Pythagorean model like Fig. 3, which is then scaled and transformed into the desired dimensions.

Furthermore, apart from the Pythagorean rectangle of the crepis outline, three more are recognized on the plan of the Apollo temple, shown in red in Fig. 2. The Naiskos, PQRS, with sides ratio 7:12, measures on the stylobate 8.358x14.328 m (Birnbaum, 2006: 161); it is therefore half of the Pythagorean rectangle 8.358x(2x14.328) m which is proportional to the Pythagorean triple (7, 24, 25). In addition, each of the rectangles GHLM and HJKL measures (20x2b)x(21x2b) and is proportional to the Pythagorean triple (20, 21, 29).

The length unit in the temple is in general considered to be the Attic foot and two alternative values are the most credible ones, either 29.85 cm (Birnbaum, 2006: 174) or 29.42 cm (Weber 2011: 45). However, a third alternative unit will be considered by the author in a forthcoming article, that is a cubit of 0.5296 m equal to one tenth of the module M and equivalent foot equal to 0.5296/1.5 = 0.353 m. It is noted that Weber's estimation of foot corresponds exactly to 0.353 x 5/6 meters and 2b/9 or M/18; it is therefore preferred as a commensurate estimation to the temple dimensions.

The geometric design and the harmonic proportions of the temple along with the roofless adyton and the axial orientation of the temple are among the outstanding features of a unique monumental architecture. Castro et al. (2016) examined five temples of Apollo on Mainland Greece and Ancient Ionia (Asia Minor), including Didyma, regarding their functioning through astronomical orientation, and showed that the rise, setting, orbit and observation of certain constellations in the celestial sphere, as well as the solar stands, can be directly related to the architecture of the temples. They underlined, that the unique architecture of the Great Temple of Apollo at Didyma, the most renowned Sanctuary and oracle after Delphi, can be related to astronomical observation.

3. INDICATIONS ON THE APPLICATION OF MATHEMATICS IN THE EUPALINOS' TUNNEL

Pythagoras was born in a period when intellectually astonishing things were happening in the neighboring city of Miletus, where Ionian natural philosophy was being developed. And on his home island Samos architectural and technical masterpieces were being realized, such as the tunnel of Eupalinos, which is still hailed as an "unsurpassed feat of engineering". This tunnel, 1,036 meters long and devised to guarantee a long-term water supply, was dug from both ends in order to shorten the construction time – a venture which required substantial mathematical and technical skills" (Riedweg, 2013: 51). Pythagoras' involvement in the design of the Eupalinos' tunnel, although reasonably suggested by Riedweg, has not been examined in this sense so far and is investigated in this article. According to Riedweg (2005: 46) the construction of the Eupalinos' tunnel "falls in Pythagoras' later youth and is hardly conceivable that he was not familiar with this bold engineering project, which must have taken years to complete".

Possible transfer of designing and monitoring expertise from the Heraion temple to the tunnel engineer cannot be excluded, since again Riedweg (2005: 45) notes that the Samian architect Theodorus who dealt with the giant temple of Hera was a many-faceted and innovative artist, who is supposed to have invented among other things a device for measuring angles, a water-level, and the lathe (Pliny, Natural History 7.198).

Certainly, a leveling device was constantly required in the construction of both, the Heraion temple and the horizontal tunnel, as well as in the positioning of the predetermined tunnel mouths. Tunneling started in parallel from both mouths, a fact meant by Herodotus' adjective "double-mouthed (αμφίστομον)" and convincingly confirmed already in 1884 by Fabricius (1884: 173-176). The tunnel floor elevation at the northern portal is 55.22 m and at the southern one 55.26 m (Kienast, 1995: Plan 2) and remains an unresolved mathematical conundrum how the one-kilometer apart portals were fixed so accurately.

The Eupalinos' aqueduct (Fig. 4) has been extensively studied and highlighted as exceptional engineering feat of the sixth century BC, as well as a mathematical problem studied already in antiquity by Heron (Burns, 1971: 173). The tunnel pierced the Kastro Hill at the same time, at two portals in the North near the Ayiades spring and in the South above the city of Samos (Fig. 5). The aqueduct is composed of three sectors, accommodating the water pipeline from the spring to the city. The supply sector, outside the city walls, carries water from the copious and still flowing spring to the northern mouth of the tunnel and the distribution sector starts from the southern mouth, within the city walls; both end sectors were excavated using the shafts-and-gallery tunneling technique (Chiotis, 2017: 5). At the interval between the end sectors, a few meters below the floor of the tunnel and simultaneously, an inclined narrow gallery was dug, on which the water line rests, comprised of interconnected terracotta pipes.

Aqueduct description is kept to a minimum in this article, given the detailed documentation by the German Archaeological Institute (Kienast, 1995), as supplemented by recent publications of independent researchers (Lyberis et al., 2014; Zambas et al., 2017; Zambas, 2017; the latter being an updated source of references), working for the project of the tunnel's restoration of the Greek Ministry of Culture. Fabricius' first study of the aqueduct in 1884 has been practically confirmed and refined by modern studies and the aqueduct is perfectly illustrated in his outstanding synthetic presentation of Fig. 4, the best concise description of the tunnel and the aqueduct.

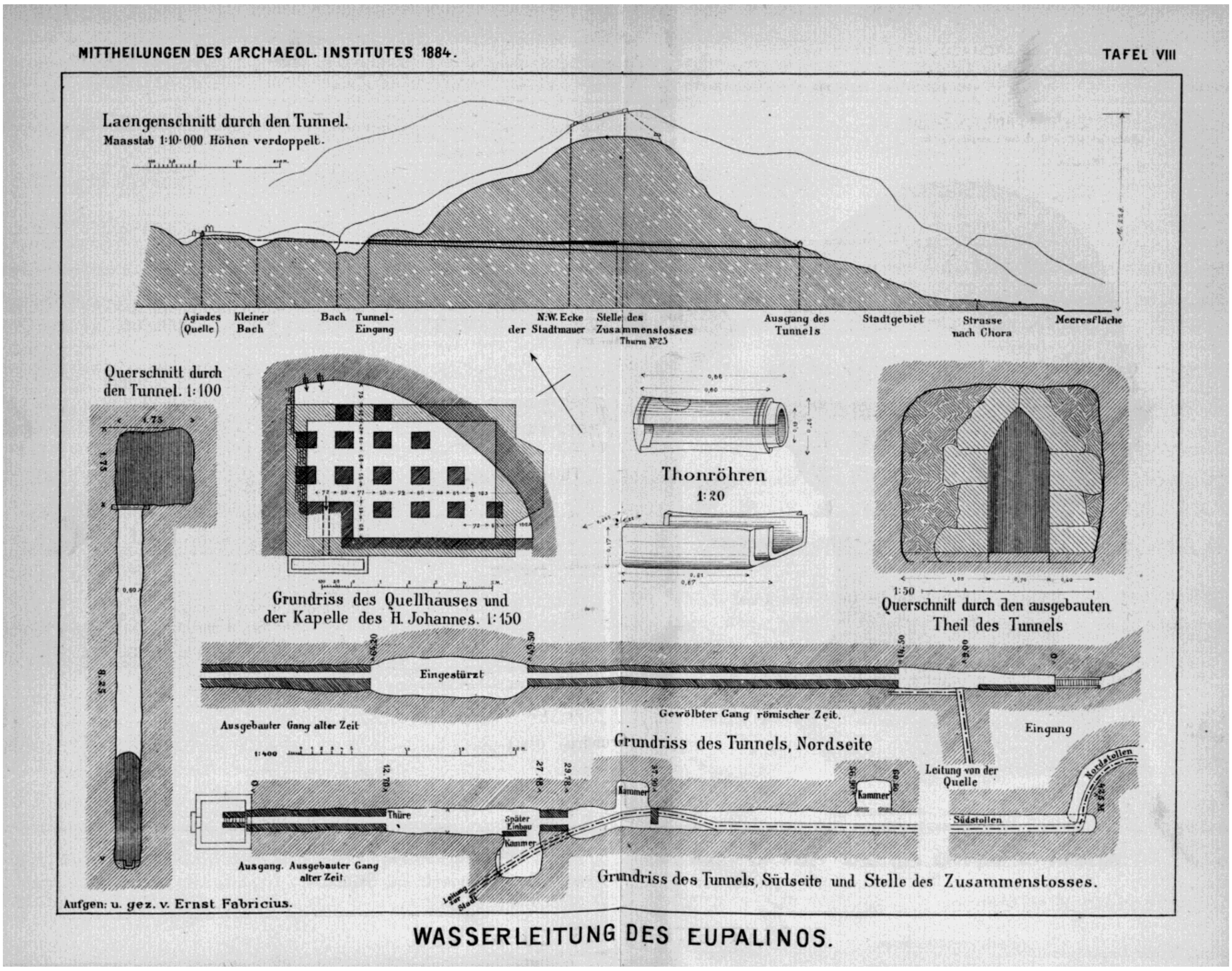

Figure 4: Eupalinos' aqueduct in general section on the top and detailed views below from Fabricius' first study in 1884.

Figure 5: Location maps projected on ESRI's satellite images: a. The aqueduct and the walls of the ancient city of Samos, in Greek coordinates (EGSA); b. The broader area, including the island Fourni where the marble for the Polycratean temple of Heraion was quarried (Cramer, 2004: 165).

3.1 Geometric drawings on a rock slab

There are hints from the Eupalinos' tunnel itself of the practical application of mathematics for the design of the tunnel. Among them a recent discovery of a slab found 132 m from the north mouth during the restoration of the Archaic lining, with an incised rough geometric drawing (Fig. 6). The meaning of this geometric construction is not obvious. It seems like a mason's explanation of a geometric construction or possibly a comment on the V-shaped deviation of the north bore of the tunnel according to Zambas (2017: 126, his Fig. 27).

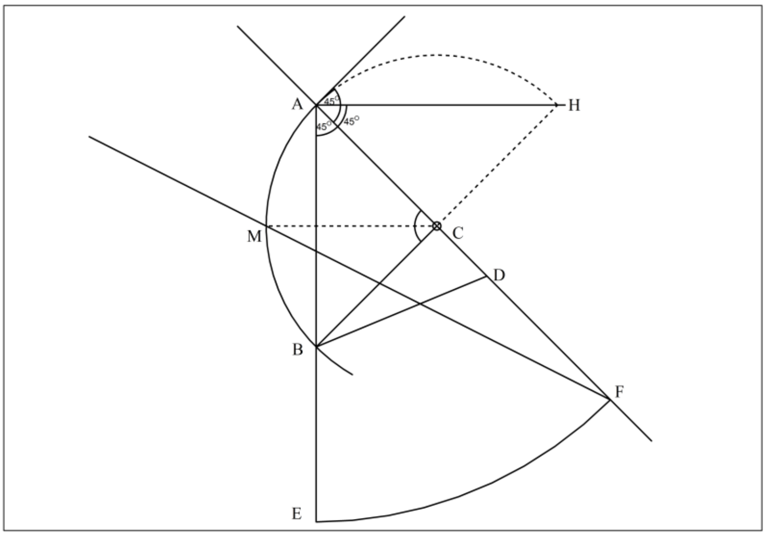

Figure 6: Our interpretation of the ancient drawing carved on a rock slab from the Eupalinos' tunnel lining, based on the slab's photo published by Zambas (2017). M is taken in the middle of the quadrant AB. Dashed lines were added to the drawing.

In our interpretation, the ancient drawing on the slab displays basic geometric rules, as if prepared for instructions by a mathematician to an engineer. The central angle ACB in a quadrant is right; the inscribed angle BAF in a quadrant is half of the right angle; the inscribed angle BAH in a semicircle is right, as expected from Thales' theorem; the tangent to the circle at A is drawn perpendicular to the radius; the right angle fractions of ¼, ½ and ¾ are also drawn and their tangents can be calculated as ratios of sides in right triangles.

3.2 The deviation from the alignment and possible application of Thales' theorem in tunnel surveying

It is generally accepted that the tunnel was planned horizontal, aligned between the predefined mouths but the original plan was significantly modified in the northern branch, and rather relatively early, given the significant deviation from the alignment about 250 meters from the north end.

Between points 23 and 24 of the longitudinal plan (Kienast, 1995: Plan 3a) of the northern branch, at a distance of c. 240 meters from the northern end (Fig. 7), there are adjacent symbols K and Λ of the ancient measurements at a distance of only 2.5 meters apart. However, the regular distance of the sequential measuring marks of the system 1 in the North is estimated by us to 20.52 m. As Kienast correctly concludes, the short distance between the symbols K and Λ indicates that the symbol Λ belongs to an earlier series of measurements from a different starting point.

We verified this conclusion through the calculation of the lengthening due to the deviation up to the point A in Fig. 7. It was found to be 18.42 m and, by the addition of 2.5 m for the distance between the points K and Λ, the interval of 20.92 m results relatively close to our estimation of the regular interval of ancient measurements of 20.52 m. In any case, depicting during tunneling the actual routing along the triangular detour and further up to the crossing point is a complicated task that requires accurate surveying measurements. Even the so-called "triangular" detour, shown in Fig. 7, is more complex than this description suggests, because the course between the points K and Σ is a crocked path and observing is hindered at least between the points 2 and 4, 6 and 8, 7 and 9, 9 and B and Σ and Π. Therefore, recording the tunnel direction is mandatory and the successful crossing infers accurate topographic mapping during tunneling for which we propose a possible method based on Thales' theorem.

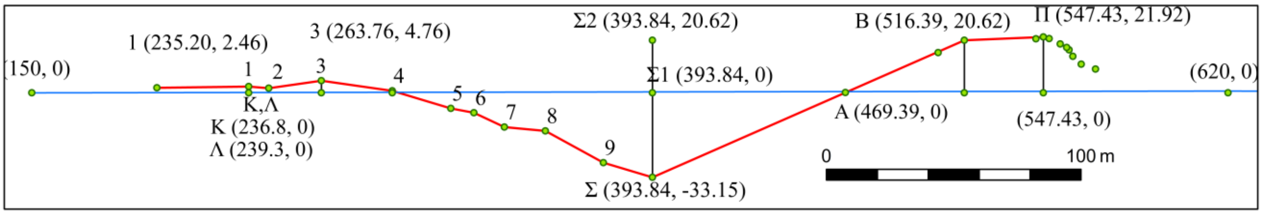

Figure 7: The route of the "triangular" detour and coordinates of critical points in regard with the tunnel axis, adopted from Kienast's longitudinal plan (1995: Plan 3a) of the northern branch. It is noted that observation between points: (2, 4), (6, 8), (7, 9), (9, B), (Σ, Π) is hindered.

Using a measuring cord or rod the direction change CAB, shown in Fig. 8, between two angular branches of the tunnel can be measured as a ratio of the perpendicular segments CB and CA, following the Egyptian practice. In the extension of the old direction it can be taken AM=MB=1, one length unit supposedly one cubit, and MC equal to one unit again to define the point C. Thus, ACB is a right angle according to Thales' theorem, since it is circumscribed in circle of diameter AB centered at M. In this way the "angle" between successive segments is measured as a ratio CB/CA sufficient for graphical solution for drawing the tunnel geometry.

Equally well the ratio AC/AB can be used which corresponds to the notion of spread in Rational Trigonometry. The spread between two lines is a dimensionless quantity, and in the rational or decimal number fields takes on values between 0 and 1, with 0 occurring when lines are parallel and 1 occurring when lines are perpendicular. Forty-five degrees becomes a spread of 1/2, while thirty and sixty degrees become respectively spreads of 1/4 and 3/4. What could be simpler than that? (Wildberger, 2005: 13).

CD can also be measured to be used for the calculation of EF, the lateral offset from the previous direction, based on the similarity of the triangles ACD and AEF. The graphical solution of a scaled drawing on a board, as suggested by Riedweg (2005: 45), seems more realistic than the geometric design in full scale on a horizontal plane on one of the extensive beaches near the ancient city as suggested by Zambas (2017: 136).

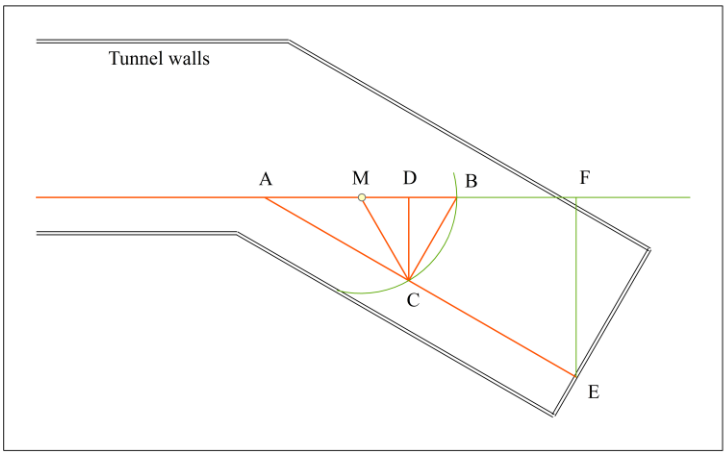

Figure 8: Possible application of Thales' theorem for the measurement of "angles" along the tunnel's crooked course.

3.3 The crossing maneuver and the stone bosses

A peculiar class of quasi measuring marks, not applied with paint and quite different from the rest ones are stone bosses protruding from the center of the gallery roof; they are up to 20 cm in height, at irregular distances from one to forty meters (Kienast, 1995: 163). Remarkably, they occur only along the meeting region of both branches, but their use and meaning are not clear.

In the northern branch there are nine bosses which lie along a smooth sigmoid path close to hearing distance from the southern branch, indicating self-reliance in the success of breakthrough and accurate surveying control. Instead of rushing to cross the southern branch along a shorter straight path, a gentle but longer maneuver was followed aiming at crossing at a right angle the deviated southern branch, as accurately as if they could observe it. This is clearly sketched by Fabricius at the lower right corner in Fig. 4, as well as in Fig. 9. We believe that this maneuver was not accidental but planned based on carefully calculated measurements and achieved by following exact tunneling instructions.

There are also five bosses of the southern branch which lie on a straight line but are not needed for keeping the alignment (Fig. 9). It is therefore questioned whether they encrypt a message. Possibly, the arrangement of the bosses B_2, B_3 and B_4 in the southern branch could indicate division according to the golden rule ratio of 1.618. The ratios B2B4/B3B4 and B3B4/B2B are approximately equal to this value of 1.618. In fact, it is measured on Kienast's longitudinal southern plan that B2B3 = 18.79 m και B3B4 = 30.35 m, so that B2B4/B3B4= 1.619 and B3B4/B2B3= 1.615. The arrangement of the bosses might be unintended, but further investigation is recommended of their enigmatic nature and function.

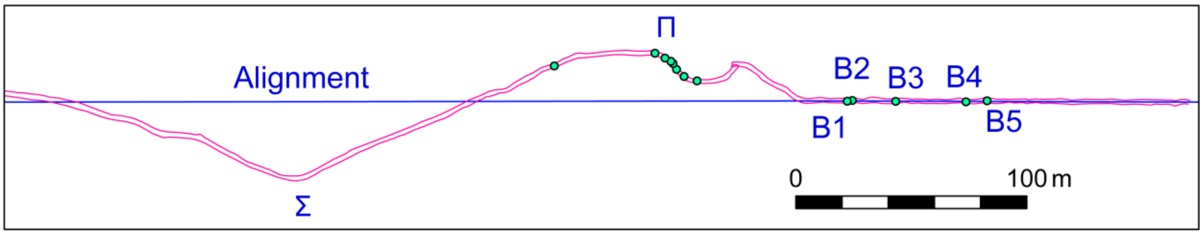

Figure 9: Stone bosses B1 to B5 along the southern branch of the tunnel reproduced from Kienast's longitudinal plan 3b.

3.4 Length unit and the tunnel length

The measuring interval of marks associated with ancient tunnel measurements described by Kienast (1995: 151 and 156), normally corresponds to 40 and 120 length units for the first and the second system of ancient measurements, respectively. However, some of the intervals are significantly longer or shorter, deviate from the above integers and because of that division of the calculated average interval by 40 or 120 for the estimation of the length unit can be misleading. We preceded to the estimation of the length unit from the measurements of the system 2, considered to be more accurate and consistent, taken after the tunnel breakthrough.

The estimation of the length unit was attempted through a statistical procedure designed especially for this case. It was based on the assumption that distances between measurement points are simply multiples of the length unit, the cubit. A deviation-error index was devised, and the length unit estimate was taken as the one that minimizes this deviation index. The distance of each pair was divided by an assumed value of length unit in the range 0.5 to 0.55 m. Then, the nearest integer to this ratio was calculated and multiplied by the assumed length unit. The actual pair distance was subtracted from this product and squared for all pairs of marks; finally, the sum of the squares was calculated, and this calculation was repeated stepwise for consecutive values of assumed cubit length. The assumed value of length unit with the minimum sum of squared differences was taken as the best estimate of length unit. The procedure is similar in principle to the cosine quantogram described by Pakkanen (2013: 16), based however directly on the measurements. In this way the value of 0.52 m was calculated for the length unit of the system 2 (Fig. 10).

It is noted that both modern tunnel measurements (Kienast, 1995: 42; Zambas 2017: 122), have common conventional zero point taken on the lowest step of the modern stair of the portal in the north. Based on the measuring marks and the measuring intervals, the zero points of measurements used in antiquity were estimated as a step for addressing the question whether the tunnel's length was already estimated before tunneling works. It was calculated that the northern zero point of measurements in antiquity was about 24 m from the tunnel mouth and about 10.8 m from the southern one.

Next, the straight distance between the zero points of the measuring marks is estimated. This distance, from the north to the south, would be 27.46+ 1002.8 +10.82 = 1041.08 m. It is remarkably close to 50×40×0.52= 1040 m, where 0.52 m is the previously estimated length unit of cubit. Unless this round value is a rare coincidence, it is envisaged that the targeted distance between the end zero points in antiquity was defined in advance equal to 2000 Samian cubits.

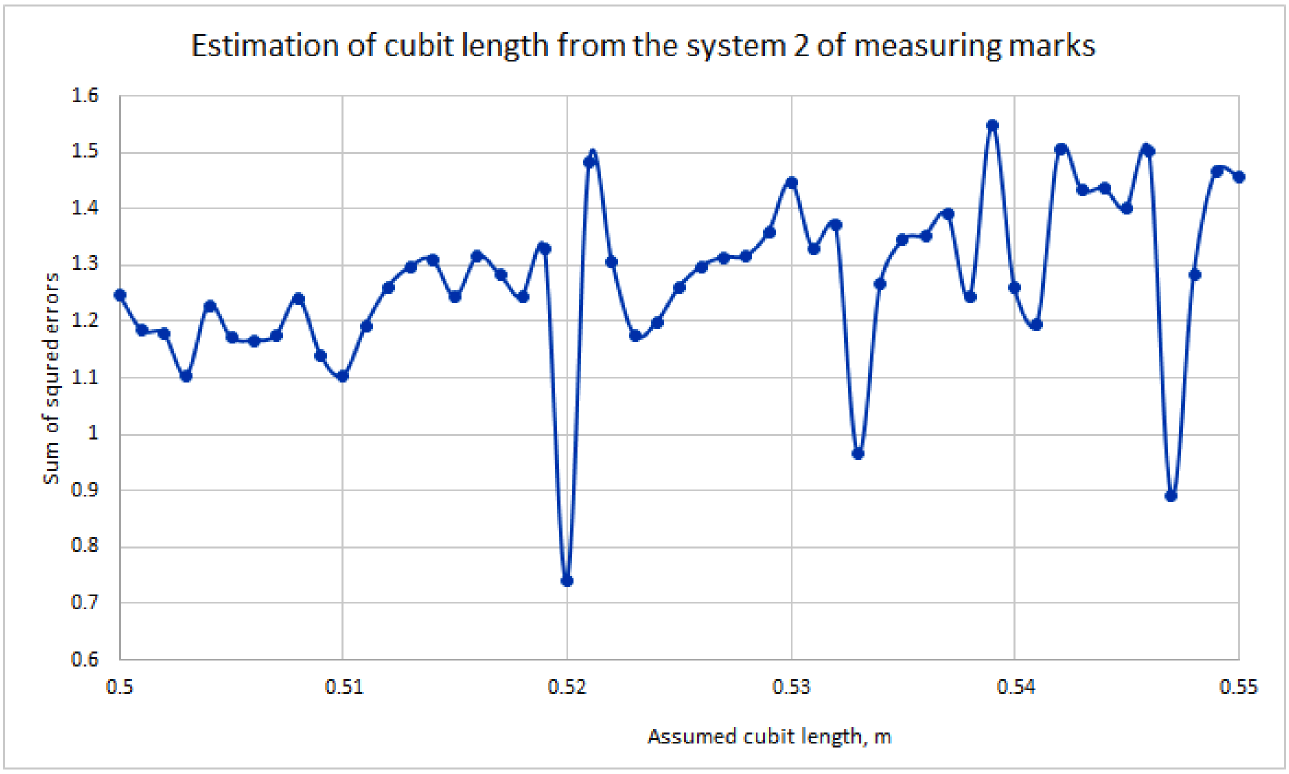

Figure 10: Diagram for the estimation of the length unit from the measuring marks of the system 2; the best estimate is taken at the points 0.52 m of minimum error index.

3.5 Possible application of the Pythagorean theorem in land surveying

The accuracy in positioning the tunnel mouths is astonishing and doubtlessly confirmed by modern surveying, but difficult to explain. It is generally envisaged that most likely tunneling was contemporary with the construction of the city walls or marginally posterior and this could have facilitated surveying. Towers of the circuit walls near the crest of the Kastro Hill for example could have been used for the alignment between the candidate sites for the mouths along a rocky profile.

On the other hand, elevation measurements could proceed along another route at a second stage, after the alignment, along smoother paths such as AN-AS or BA-BS as shown in Fig. 11. Defining the level independently of the alignment has been also suggested by Rihll and Tucker (2003: 411). It would suffice to measure the elevation difference between the mouth sites N and S and a third convenient point like A or B. Along these paths the Kastro crest is bypassed, the elevation difference is smaller and the topography is smoother. Furthermore, the tunnel length could also have been calculated based on the length measurement of a shorter interval, like the perpendiculars AA' or BB' to the tunnel alignment. Accurate length measurements would be convenient by scaffolding. By the application of the Pythagorean theorem in combination with similar triangles the tunnel length and the elevation difference at N and S could have been calculated. After all, the successful breakthrough of the tunnel through the "triangular" deviation indicates the ability of surveying along slalom routing. No doubt, the achieved accuracy in positioning the mouths in advance and the well-designed maneuver at the crossing point indicate the involvement of a mathematical mind supporting Eupalinos' engineering skills.

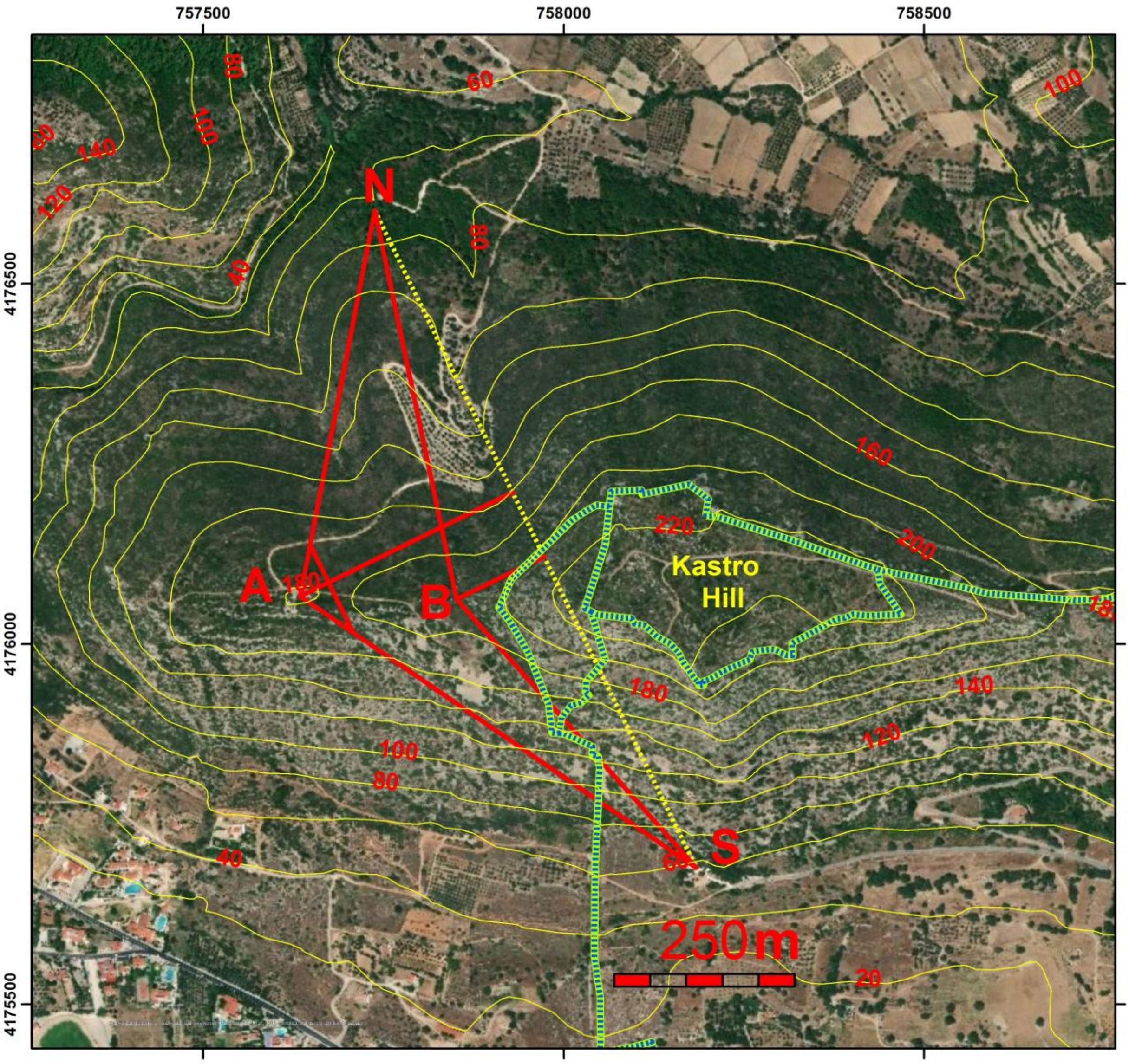

Figure 11: Proposed method of surface surveying for measuring AA΄ or BB΄, angle φ and elevation difference between the tunnel ends N and S and convenient points A or B, projected on ESRI's satellite images in Greek coordinates (EGSA).

4. DISCUSSION

4.1 On the temples' design

From the preceding analysis on the architectural layout of Late Archaic temples, the systematic application of a broad variety the Pythagorean triples is confirmed as a basic design tool both in Samos and Magna Graecia, accomplished in periods of Pythagoras' influential presence there. The procedure was fully developed at the Heraion temple of Samos in the sixth century BC and the following basic steps are recognized.

1. Selection of a cardinal Pythagorean rectangle as a model for the temple outline, which could be either the crepis or the stylobate or the colonnade; in the latter case the intercolumnar axes or the outer colonnade can alternatively be used, as at the Heraion temple.

2. Selection of a modulus for designing a grid, square or rectangular, related mostly with the interaxial column interval.

3. Subdivision of the model plan into smaller either Pythagorean or harmonic rectangles.

4. Correlation of the plan proportions to the elevation by extending the plan grid vertically or through a modified vertical grid on another Pythagorean rectangle. In this way the building proportions are interconnected in three dimensions.

5. Calculation of the actual dimensions from the model, based on a scale factor.

6. Accurate positioning of the Pythagorean rectangles constrained by the dimensions of the sides and the diagonals, along with implementation of the grid and delineation of architectural elements in situ.

The described basic method was fully developed in Pythagoras' time and perfected in the Hellenistic times, when more emphasis was put perhaps on the harmonic proportions.

4.2 On the Pythagorean triples in Babylonian mathematics

Neugebauer and Sachs (1945) deciphered and revealed the importance of mathematical Babylonian cuneiform tablets and concluded that the Plimpton 322 tablet, dated in the early second millennium BC, listed Pythagorean triples. More specifically, according to Neugebauer (1951: 40) there is a strong indication that the fundamental formula for the construction of triples of Pythagorean numbers was known to the Babylonians.

The tablet was originally larger, it was broken, and four columns of numbers are only preserved. In the second and third columns the numbers are Pythagorean, integer solutions b and d of the equation:

$d^2 = b^2 + l^2$

whereas the number of the fourth column corresponds to $\frac{d^2}{l^2}$, where d the hypotenuse and *l* the long leg.

Neugebauer obtained the Pythagorean triples (a, b, c) of the tablet from the generating functions:

$a = p^2 + q^2$, $b = p^2 - q^2$ and $c=2pq$

where p and q are arbitrary integers subject only to the condition that they are relatively prime, not simultaneously odd and p > q. Neugebauer (1957: 42) assumed that "this is indeed the formula which we needed for our explanation of the text dealing with Pythagorean numbers". However, this is Euclid's approach for the generation of Pythagorean triples, introduced much later.

Neugebauer and Sachs' views were disputed soon by Bruins (1949: 629) who proved that a simpler interpretation is possible, in which the production of Pythagorean numbers is feasible by using only one parameter, instead of the couple (p, q) of independent integers, by means of reciprocal sexagesimal numbers derived from Babylonian tablets.

Friberg (1981: 284) verified that the values listed in the Plimpton 322 tablet are precisely the ones that can be obtained from reciprocal pairs, under the condition that the reciprocal numbers t and t' are "regular", that is in the form: $t = 2^{\alpha}3^{\beta}5^{\gamma}$ where α, β, γ are integers not necessarily positive. Friberg went further to generate an arbitrarily large set of admissible values t, by letting the parameter t and its reciprocal t′ as t=s/r and t′=r/s vary within a bounded strip in the (r, s) plane. So, Friberg, like Neugebauer, envisaged in the tablet "anachronistic" mathematics supposedly to be known by the Babylonians.

To clarify this point further and make this discrepancy better understood let us refer to the tablet YBC 6967 the calculations of which fortunately are described in the tablet. Høyrup (1990: 262-266) interpreted the impressive underlying "cut-and-paste" or "naive" geometric methodology on the solution of the system of equations:

xy=60 and x-y = 7.

The problem deals with a pair of numbers (12 and 5, members of the Pythagorean triple 5, 12, 13) and the solution is given by a clever geometrical interpretation; any modernizing algebraic solution would be therefore irrelevant and out of historical context.

Plimpton 322 tablet has been and continues to the be subject of intensive and multidisciplinary research, but a few references only closely related to our topic, are compiled here. Robson (2001: 167) compared and evaluated in a broader mathematico-historical context both alternative interpretations, Neugebauer's proposal of generating functions with two parameters and Bruins' approach based on one parameter and reciprocal sexagesimal numbers from tablets. She based her judgement on certain criteria, the first of which was the historical sensitivity and the condition that "the theory should respect the historical context of Plimpton 322 and not impose conceptually anachronistic interpretations on it" (Robson, 2001: 176). She considered the first column in decimal notation as the ratio d^2/l^2 or b^2/l^2, depending on the acceptance or not of the supposed missing unit of the broken part of the tablet, where *d* is the hypotenuse, *l* the long side and *b* the short one. She also transliterated a grammatically and mathematically meaningful heading for Column I, as "The takiltum of the diagonal from which 1 is torn out, so that the short side…".

We believe that this heading, as translated above, is a concise expression of the Babylonian "diagonal rule", the Pythagorean theorem in modern terminology, transliterated in our algebraic notation as:

$d^2/l^2-1= b^2/l^2$

a genuinely beautiful equation in normalized notation.

Robson (2001: 167) showed that the Neugebauer's widespread theory of generating functions cannot be correct. She provided supporting evidence for an alternative way of triples generation using regular reciprocal pairs and applying common Babylonian mathematics. She also proposed a possible completion of the 15 rows of the tablet with the missing columns (Robson, 2001: 185-186).

As to the purpose of the tablet, Robson's remarks are enlightening (2002: 118): "Plimpton 322, analyzed solely as a piece of mathematics, looked very modern, millennia ahead of its time, incomparably more sophisticated than other ancient mathematical documents. But if we treat Plimpton 322 as a cuneiform tablet that just happens to have mathematics on it, a very different picture emerges. We see that it is a product of a very particular place and time, heavily dependent on the ancient scribal environment for its physical layout as a table, its mathematical content, and its function as a teacher's aid. All the techniques it uses are widely attested elsewhere in the corpus of ancient Mesopotamian school mathematics. In this light we can admire the organizational and arithmetical skills of its ancient author but can no longer treat him as a far-sighted genius. Any resemblance Plimpton 322 might bear to modern mathematics is in our minds, not his". Incidentally, according to Robson (2002: 111), the tablet was written by someone familiar with the temple administration in the Mesopotamian city of Larsa in around 1800 BC.

In our opinion, the unique Plimpton 322 tablet could be of practical significance too, since the triples offer a good basis for the design of Pythagorean rectangles, useful for the layout of grids in architecture and the subdivision of land parcels, as well as for the layout of inclined surfaces.

4.3 On the Pythagorean triples of the temples examined

We continue with the investigation of the generation method of the Pythagorean triples of the temples examined, starting as usual, from the Pythagorean equation with a<b<c:

$a^2+b^2 = c^2$ and for a=1 it is:

$1= c^2-b^2 = (c+b)(c-b)$, $c+b = \lambda$ and $c-b = 1/\lambda$.

$b=\frac{1}{2}(\lambda - 1/\lambda)$ and $c=\frac{1}{2}(\lambda + 1/\lambda)$.

For a supposed Pythagorean triple A<B<C, λ can be calculated from either of the equations $\lambda^2 -2\frac{B}{A}\lambda - 1 =0$ and/or $\lambda^2 -2\frac{C}{A}\lambda + 1 =0$

If λ can be expressed as a fraction of integers R_1 and R_2, then

$\lambda = \frac{R1}{R2}$ *and* $b = \frac{\lambda^2-1}{2\lambda} = \frac{\frac{R1}{R2}-\frac{R2}{R1}}{2}$ and $c = \frac{\lambda^2+1}{2\lambda} = \frac{\frac{R1}{R2}+\frac{R2}{R1}}{2}$

The reduced triad of rational numbers:

$\frac{1}{2}\{\frac{R1}{R2}+\frac{R2}{R1}\}$, $\frac{1}{2}\{\frac{R1}{R2}-\frac{R2}{R1}\}$ and 1

satisfies the Pythagorean equation because:

$\frac{1}{4}\{\frac{R1}{R2}+\frac{R2}{R1}\}^2 = \frac{1}{4}\{\frac{R1}{R2}-\frac{R2}{R1}\}^2 + 1$

Incidentally, this is the algebraic expression of a Babylonian algorithm proven by cut-and-paste by Simoson (2019).

Then $A=R_1R_2$, $B=bA$ and $C=cA$, where:

$B=\frac{1}{2}\{\frac{R1}{R2}-\frac{R2}{R1}\}R_1R_2 = \frac{1}{2}\{(R_1)^2-(R_2)^2\}$

$C=\frac{1}{2}\{\frac{R1}{R2}+\frac{R2}{R1}\}R_1R_2 = \frac{1}{2}\{(R_1)^2+(R_2)^2\}$

So, A, B, C make up a Pythagorean triple because R_1, R_2, A, B and C are integers and $A^2+B^2=C^2$.

The above equations are actually Euclid's formulas and can be used for the calculation of Pythagorean triples for an arbitrary pair of integers (R_1, R_2). They were applied to the triples calculated at the examined temples and the relevant coefficients c and b are shown in the Table 3. It is found that if either R_1 or R_2 is an even integer, then $A=2R_1R_2$. It is worth noting that both methods, the Euclid's formulas and the simpler approach of reciprocal pairs produce identical results. It is therefore concluded that Neugebauer's persistence on the advanced formulas of generating functions is an unnecessary anachronism.

Table 3: Validation of the Pythagorean triples of the ancient temples investigated.

Pythagorean triple (A, B, C)	Temple	$\lambda=\frac{R1}{R2}$	(R1, R2)	$c = \frac{\frac{R1}{R2}+\frac{R2}{R1}}{2}$	$b = \frac{\frac{R1}{R2}-\frac{R2}{R1}}{2}$
115, 252, 277	Heraion	23/5	(23, 5)	2.408696	2.191304
105, 208, 233	Heraion	21/5	(21, 5)	2.219048	1.808696
48, 55, 73	Heraion	8/3	(8, 3)	1.520833	1.145833
5,12,13	Heraion Athena, Paestum	5	(5, 1)	2.6	2.4
8,15,17	Trapeza, Aigialeia	4	(4, 1)	2.125	1.875
7,24,25	Athena, Paestum Didyma	7	(7, 1)	3.571429	3.428571
20,21,29	Didyma	5/2	(5, 2)	1.45	1.05
16,63,65	Didyma	8	(8, 2)	4.0625	3.9375

The Pythagorean triples of the temples in Table 3 are not included in the fifteen triples of the Plimpton 322 tablet and only three of them - (5, 12, 13), (7, 24, 25) and (8, 15, 17) - are among the 38 triples of the extended version calculated by Simoson (2019). This can be considered as a strong indication that the Late Archaic Pythagorean triples in the Greek temples were produced independently and did not originate from the Plimpton 322, the unique known tablet with Pythagorean triples as remarked by Robson (2002: 108).

4.4 Hints on Pythagoras' contribution to the field of mathematics

In recent scholarship the consensus view on the Pythagorean theorem has received strong challenges, which in agreement with Neugebauer's views are best exemplified in the Stanford Encyclopedia of Philosophy (2018), summarized as following. "There is evidence that Pythagoras valued relationships between numbers such as those embodied in the so-called Pythagorean theorem, though it is not likely that he proved the theorem. All that tradition ascribes to Pythagoras, then, is discovery of the truth contained in the theorem. The truth may not have been in general form but rather focused on the simplest such triangle (with sides 3, 4 and 5), pointing out that such a triangle and all others like it will have a right angle. Modern scholarship has shown, moreover, that long before Pythagoras the Babylonians were aware of the basic Pythagorean rule and could generate Pythagorean triples, although they never formulated the theorem in explicit form or proved it. Thus, it is likely that Pythagoras and other Greeks first encountered the truth of the theorem as a Babylonian arithmetical technique. It is possible, then, that Pythagoras just passed on to the Greeks a truth that he learned from the East. All that this tradition ascribes to Pythagoras, then, is discovery of the truth contained in the theorem. The truth may not have been in general form but rather focused on the simplest such triangle (with sides 3, 4 and 5), pointing out that such a triangle and all others like it will have a right angle. What emerges from this evidence, then, is not Pythagoras as the master geometer, who provides rigorous proofs, but rather Pythagoras as someone who recognizes and celebrates certain geometrical relationships as of high importance".

As expected, Neugebauer was fully aware of the level of the Babylonian mathematics when writing that "in spite of the numerical and algebraic skill and in spite of the abstract interest which is conspicuous in so many examples, the contents of Babylonian mathematics remained profoundly elementary. Babylonian mathematics never transgressed the threshold of prescientific thought. It is only in the last three centuries of Babylonian history and in the field of mathematical astronomy that the Babylonian mathematicians or astronomers reached parity with their Greek contemporaries" (Neugebauer, 1957: 48).

However, unjustifiably, he degraded the contribution of early Greek philosophers in mathematics, as inferred from his comments (1957: 148, 149, 152).

- "It seems to me evident, however, that the traditional stories of discoveries made by Thales or Pythagoras must be discarded as totally unhistorical".
- "The elementary theory of numbers, however, may or may not eventually be based on much older oriental material. I do not doubt that any connection with the name of Pythagoras is purely legendary and of no historical value".
- "I think that it is evident that Plato's role has been widely exaggerated. His own direct contributions to mathematical knowledge were obviously nil".

It is commonly repeated that Pythagoras' theorem was already known in Mesopotamia in 1500 BC and Leonid Zhmud (2003) meaningfully notes in his review of Riedweg's book "*Pythagoras. Leben, Lehre, Nachwirkung*" that Riedweg (2002) mentions this twice. Nevertheless, Zhmud convincingly remarks that in fact, what the Babylonians knew was not a general geometrical proposition, let alone its deductive proof, but only an empirical arithmetic formula for some Pythagorean triples (i.e. 3, 4, 5; 5, 12, 13, etc.)".

Even more enlightening on that is Burkert (1972: 401), in his monumental book, in a section entitled "Did the Pythagoreans lay the foundations of Greek mathematics?" he notes that "as pre-Greek mathematics has been rediscovered in Egyptian papyri and Babylonian clay tablets, a clearer light has been thrown on the outstanding achievement of the Greeks in the development of pure mathematics. The Babylonians had made considerable progress in the accumulation of detailed knowledge, in practical calculation, and in the solution of even rather complicated problems in arithmetic; beyond question, the Greeks had much to learn from them. But it was always single problems they were concerned with, making use of certain "recipes," without any theoretical explanation or even an attempt at proof; we cannot even be certain that the Babylonians formulated theorems in general terms. Some of the "recipes" or formulas are inexact, but this did not matter as long as they provided a practically useful approximation. Only with the advent of Greek geometry do we find the demand for generalized and stringent proof, for a deductive system based on axioms and postulates. This is the system presented to us in the Elements of Euclid, model which until the nineteenth century seemed not to require any essential improvement. All later achievements, including those of the Indians and the Arabs, build on the foundations laid by the Greeks".

On the query "Who discovered the Pythagorean theorem?" Meera Nanda (2016: 47) concluded that: "the geometric relationship described by this theorem was discovered independently in many ancient civilizations. The likely explanation is that the knowledge of the relationship between sides of a right-angle triangle emerged out of practical problems that all civilizations necessarily face, namely, land measurement and construction of buildings – buildings as intricate as the Vedic fire altars, as grand as the Pyramids, as functional as the Chinese dams and bridges, or as humble as simple dwellings with walls perpendicular to the floor". As summarized by Nanda (2016: 21) "The first recorded evidence for the Pythagorean conjecture dates back to some 1800 years BCE and it comes from Mesopotamia, the present-day Iraq. The first proof comes from the Chinese, preempting the Euclidean proof by a couple of centuries, and the Indian proof by at least 1000 years. Even though Pythagoras was not the first to discover and prove this theorem, it does not diminish his achievement. He remains an extremely influential figure not just for history of mathematics, but history of science as well. Pythagoras and his followers were the "first theorists to have attempted deliberately to give the knowledge of nature a quantitative, mathematical foundation". Giants of the Scientific Revolution, including Johannes Kepler and Galileo Galilei walked in the footsteps of Pythagoras.

However, the gap between a practical rule and a theorem is huge and Exarchakos (2006: 92) is right to remark that there is no theoretical approach in the Babylonian mathematics, nor a general proposal proven on logical reasoning to be considered as a theorem. We believe therefore that what was discovered in many ancient civilizations was a practical rule, the diagonal rule in the case of the Babylonians, but not a theorem embodied in a general theoretical system.

On this point Angelika-Nikita (2018: 61) remarks that" The Greeks understood something that had somehow eluded the Egyptians and Babylonians: the importance of mathematical rigor. Rigor was the thoroughness and attention to detail for improving accuracy. For example, ancient Egyptians, equated the area of a circle to the area of a square with sides equal to 8/9 of the circle's diameter. According to this calculation, the value of the mathematical constant π is 256/81. Though it is a highly accurate calculation (around 0.5% error), it is mathematically incorrect. However, for the purposes of Egyptian engineering, this error was insignificant. But, ignoring this 0.5% error neglects a fundamental property of the true value of π, that no fraction can express it, as it is an irrational number".

The real – and path-breaking – contribution of Pythagoras was the fundamental idea that nature can be understood through mathematics. He was the first to imagine the cosmos as an ordered and harmonious whole, whose laws could be understood by understanding the ratios and proportions between the constituents. It was this tradition that was embraced by Plato, and through Plato became a part of Western Christianity, and later became a fundamental belief of the Scientific Revolution expressed eloquently by Galileo: "The Book of Nature is written in the language of mathematics" (Nanda, 2016: 33).

It is similarly underlined by Burov and Burov (2015) that when Galileo stated this, he was expressing the ancient Pythagorean credo. The same can be said about Dirac, whose fundamental belief was that "the laws of nature should be expressed in beautiful equations". Our universe is special not only because it is populated by living and conscious beings but also because it is theoretizable by means of elegant mathematical forms, both rather simple in presentation and extremely rich in consequences. Such a special universe deserves a proper term, and we do not see a better choice than to call it Cosmos or to qualify it as Pythagorean, in honor of the first prophet of theoretical cognition, who coined such important words as cosmos (order), philosophy (love of wisdom), and theory (contemplation).

5. EPILOGUE

Although Pythagoras' involvement in the design of temples cannot be directly proved, it is a reasonable assumption, given the crucial and innovative application of the Pythagorean triples at the Heraion temple during his period in Samos. The foundation of the second peripteros temple of Heraion started during Pythagoras' time in Samos. The early application of the Pythagorean triples is revealed in the architectural design of the huge temple, based on a mathematical model thanks to which a coherent system of proportions was realized. For geometrical accuracy, a grid was used, and dimensioning was based on a common module for the various parts of the monument. The Heraion temple resembles a demonstration project of the application of Pythagorean triples which are not related to the triples of the Old Babylonian Plimpton 322 tablet.

A Pythagorean rectangle was also recognized at the Trapeza temple, built in Pythagoras' time at the Achaean Metropolis of Croton. The same process of design based on Pythagorean triples, in particular in three dimensions, is also confirmed in the Athena temple at Paestum during the period of the greatest philosophical influence of Pythagoras in Magna Graecia. His influence here was dual, geometric, and aesthetic, thanks to the combined application of mathematics and harmonic proportions inspired from Pythagoras' philosophy. The dual geometric-harmonic design of temples was fully developed in Pythagoras' time, starting from the Heraion temple in Samos, and his philosophy of proportions, amalgamated perhaps with Platonic ideas, prevailed in later times until the present.

The Eupalinos' aqueduct is a revolutionary work in many aspects, accomplished while Pythagoras' was in Samos. The surveying problems of the tunnel were examined in this article, to evaluate the level of mathematics involved. The complications of the work and the astonishing accuracy achieved would have been impossible without the application of mathematics and proper instrumentation; in this regard, it is reminded that the Architect Theodorus of Samos is credited with the invention of several measuring instruments. It is demonstrated that in addition to the Eupalinos' engineering skills a mathematical mind was required for the accomplishment of the work.

As to the Plimpton 322 tablet, it is concluded that the method of reciprocal pairs is the most convenient for the generation of the Pythagorean triples. Besides the tablet application in teaching, a practical use is also envisaged. It is worth noting that both methods, the Euclid's formulas and the simpler method of reciprocal pairs produce identical results. It is therefore concluded that Neugebauer's persistence on the advanced formulas of generating functions is an unnecessary anachronism.

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